

ON THE FULL CALCULUS OF PSEUDO-DIFFERENTIAL OPERATORS ON BOUNDARY GROUPOIDS WITH POLYNOMIAL GROWTH

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ABSTRACT. In this paper, we enlarge the space of uniformly supported pseudo-differential operators on some groupoids by considering kernels satisfying certain asymptotic estimates. We show that such enlarged space contains the compact parametrix, and the generalized inverse of uniformly supported operators with Fredholm vector representation.

1. INTRODUCTION

In this article, we estimate the kernel of generalized inverse of Fredholm Elliptic operators defined on boundary groupoids.

Our work is motivated by [17], which is in turn motivated by the study of differential operators on manifolds with boundary [11], [10].

Recall that in the classical construction, one first fixes a boundary defining function ρ , a smooth non-negative function on M with non-zero derivative on the boundary ∂M . Then an open neighborhood of $\partial M \subset M$ is identified with $[0, 1) \times \partial M$ (with $[0, 1)$ parameterized by ρ). Differential operators tangential to the boundary are written in the form $\rho \partial_\rho + \dots$, and can be identified with kernels on the blowup M_b , known as the b -stretched product. The b -stretched product has three boundary defining functions ρ_{01}, ρ_{10} and ρ_{11} . By some explicit calculations, it can be shown that the generalized inverse of a Fredholm elliptic operator is a kernel with asymptotic expansion in $\rho_{01}, \rho_{10}, \rho_{11}$. The space of such kernels is known as the full calculus.

In the example of, say, natural differential operators on Poisson manifolds, however, there is no obvious notion of boundary defining functions. Instead, one uses the theory groupoid (pseudo)-differential operators to characterize natural operators.

The notion of pseudo-differential operators on a groupoid was first introduced by Nistor, Weinstein and Xu [14]. It was further developed by Ammann et. al. into so called Lie manifolds, or manifolds with Lie structure at infinity [2, 6]. The same authors then applied the theory to the example of polyhedral domains [1, 3, 15]. In particular, [15] considers inverse of differential operator as an element in the abstract C^* -algebra of the underlying groupoid. However, these examples are quite similar to the manifold with boundary case.

In contrast, in [17], the author takes a more geometric approach. The groupoid is taken as the fundamental object, and one attempts to do computations without explicitly referring the singular structure. The idea was applied to the example of the

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symplectic groupoid of the Bruhat sphere, where it was shown that the parametrix of an elliptic, uniformly supported pseudo-differential operator is given by a groupoid pseudo-differential with exponentially decaying kernel.

Our main objective is to generalize the result of [17] to other similar groupoids, and also describe the generalized inverse of Fredholm operators. As far as we know, this paper is the first systematic study on non-uniformly supported groupoid pseudo-differential operators in some generality, besides the purely abstract C^* -algebra construction in [6, 15]. Moreover, our work should clarify the role of the boundary defining function in these works, as well as the classical construction.

1.1. An overview of our approach. While the technical details are tedious and elementary, the idea behind our construction is actually very simple.

In Section 2, we recall some basic notions of pseudo-differential operators on a groupoid as in [14]. Then we define the notion of boundary groupoids. Essentially these groupoids are just b -stretched products with possibly non-commutative isotropy subgroups and more degenerate Lie algebroids.

In Section 3, we begin with an elementary estimate. Perhaps what is remarkable is that such estimate has no direct analogue in the classical construction. Then we write down the definition of the calculus with bounds. These are just kernels that decays exponentially on the \mathbf{s} -fiber and polynomially near the singular set, with respect to some rather arbitrarily chosen functions. We show that the convolution product respects the filtration of the calculus with bounds.

In Section 4, we describe the generalized inverse of elliptic differential operators (or uniformly supported pseudo-differential operators). Our construction is parallel to that of [10].

Given an elliptic, uniformly supported pseudo-differential operator $\Psi = \{\Psi_x\}_{x \in M}$ such that the vector representation of Ψ is Fredholm, one starts with the invariant sub-manifold \mathcal{G}_r with the lowest dimension. In that case $\Psi|_{\mathcal{G}_r}$ is an ordinary pseudo-differential operator on the manifold with bounded geometry $M_r \times G_r$ that is invertible. Therefore the result of Shubin [16] applies and $\Psi|_{\mathcal{G}_r}^{-1}$ is a kernel with exponential decay.

The second step is to extend the off-diagonal part of $\Psi|_{\mathcal{G}_r}^{-1}$ into a kernel on \mathcal{G} . In the case $\mathcal{G} = M_0 \times M_0 \sqcup G \times M_1 \times M_1$, this is constructed by taking exponential coordinates patches defined by Nistor et. al. [13] and then extend along coordinates curves. The detail of the construction is given in Appendix B. Then a uniformly supported parametrix Φ of Ψ on \mathcal{G} can be modified, so that $R := I - \Psi\Phi$ vanishes on \mathcal{G}_r .

The third step is to improve the parametrix by considering the Neumann series. One gets a parametrix up to error decaying at order $-\infty$ at the singular set. The same arguments can be repeated and the case for general r can be proved by induction on r . In the last step of the induction, one obtains the generalized inverse.

In Section 5, we give some more remarks and highlight some open problems.

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2. PRELIMINARY DEFINITIONS

2.1. Uniformly supported pseudo-differential calculus on a Lie groupoid.

In this section, we recall the standard theory of pseudo-differential calculus developed by Nistor, Weinstein and Xu [14].

Definition 2.1. A Lie groupoid $\mathcal{G} \rightrightarrows M$ consists of:

- (i) Manifolds \mathcal{G} and M ;
- (ii) A unit inclusion $\mathbf{u} : M \rightarrow \mathcal{G}$;
- (iii) Submersions $\mathbf{s}, \mathbf{t} : \mathcal{G} \rightarrow M$, called the source and target map respectively, satisfying

$$\mathbf{s} \circ \mathbf{u} = \text{id}_M = \mathbf{t} \circ \mathbf{u};$$

- (iv) A multiplication map $\mathbf{m} : \{(a, b) \in \mathcal{G} \times \mathcal{G} : \mathbf{s}(a) = \mathbf{t}(b)\} \rightarrow \mathcal{G}$, $(a, b) \mapsto ab$ that is associative and satisfies

$$\mathbf{s}(ab) = \mathbf{s}(b), \quad \mathbf{t}(ab) = \mathbf{t}(a), \quad a(\mathbf{u} \circ \mathbf{s}(a)) = a = (\mathbf{u} \circ \mathbf{t}(a))a;$$

- (v) An inverse diffeomorphism $\mathbf{i} : \mathcal{G} \rightarrow \mathcal{G}$, $a \mapsto a^{-1}$, such that $\mathbf{s}(a^{-1}) = \mathbf{t}(a)$, $\mathbf{t}(a^{-1}) = \mathbf{s}(a)$ and

$$aa^{-1} = \mathbf{u}(\mathbf{t}(a)), \quad a^{-1}a = \mathbf{u}(\mathbf{s}(a)).$$

Our definition follows the convention of [9], but with the source and target maps denoted by \mathbf{s} and \mathbf{t} instead of α and β .

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid with M compact. Fix a metric g_A on A . For each $x \in M$, a Riemannian metric on \mathcal{G}_x is defined by right invariance. Denote the family of Riemannian volume measure on \mathcal{G}_x , $x \in M$ by μ_x .

Observe that for each $x \in M$, \mathcal{G}_x is a manifold with bounded geometry (see Appendix). Therefore, \mathcal{G}_x has at most exponential volume growth.

Definition 2.2. We say that \mathcal{G} is of polynomial (volume) growth if there exists $N \in \mathbb{N}, C > 0$ such that

$$\int_{B_{g_A}(a, r)} \mu_x(b) \leq Cr^N,$$

for any ball on \mathcal{G}_x centered at $a \in \mathcal{G}_x$ with radius r .

From now on, we shall always assume that the groupoid \mathcal{G} under consideration is of polynomial growth.

Definition 2.3. A pseudo-differential operator Ψ on a groupoid \mathcal{G} of order $\leq m$ is a smooth family of pseudo-differential operators $\{\Psi_x\}_{x \in M}$, where $\Psi_x \in \Psi^m(\mathcal{G}_x)$, and satisfies the right invariance property

$$\Psi_{\mathbf{s}(a)}(\mathbf{r}_a^* f) = \mathbf{r}_a^* \Psi_{\mathbf{t}(a)}(f), \quad \forall a \in \mathcal{G}, f \in C_c^\infty(\mathcal{G}_{\mathbf{s}(a)}).$$

If, in addition, all Ψ_x are classical of order m , then we say that Ψ is classical of order m .

For a pseudo-differential operator $\Psi = \{\Psi_x\}$ on \mathcal{G} . The support of Ψ is defined to be

$$\text{Supp}(\Psi) = \overline{\bigcup_{x \in M} \text{Supp}(\Psi_x)}.$$

The operator Ψ is called uniformly supported if the set

$$\{ab^{-1} : (a, b) \in \text{Supp}(\Psi)\}$$

is a compact subset of \mathcal{G} . We denote the algebra of uniformly supported classical pseudo-differential operator of order m on \mathcal{G} by $\Psi^{[m]}(\mathcal{G})$ ($\Psi^{[m]}(\mathcal{G}, E)$ for operators defined on a sections of a vector bundle $E \rightarrow M$), and $\Psi^\bullet := \bigcup_{m \in \mathbb{Z}} \Psi^{[m]}(\mathcal{G})$.

The convolution product on \mathcal{G} is the binary operator on $C^\infty(\mathcal{G})$:

$$(1) \quad f \circ g(a) := \int_{s^{-1}(s(a))} f(ab^{-1})g(b)\mu_{s(a)}(b), \quad \forall a \in \mathcal{G}$$

for any $f, g \in C^\infty(\mathcal{G})$, provided the integral is finite for all $x \in M$.

For any $\Psi = \{\Psi_x\}_{x \in M} \in \Psi^\infty(\mathcal{G})$. The *reduced kernel* of Ψ is defined to be the distribution

$$\psi(f) := \int_M \mathbf{u}^*(\Psi(\mathbf{i}^* f))(x) \mu_M(x), \quad f \in C_c^\infty(\mathcal{G}).$$

Lemma 2.4. [14, Corollary 1] *For any $\Psi \in \Psi^\bullet(\mathcal{G})$, the reduced kernel is co-normal at M and smooth elsewhere. Moreover, the map $\Psi \mapsto \psi$, where ψ is the reduced kernel of Ψ , is an algebra isomorphism.*

Remark 2.5. By virtue of Lemma 2.4, there are three equivalent ways to define the algebra of pseudo-differential operator on \mathcal{G} , namely

- (i) Fiberwise composition $\Psi\Phi = \{\Psi_x\Phi_x\}_{x \in M}$;
- (ii) Convolution product $\psi \circ \varphi$, where ψ and φ is the reduced kernel of Ψ and Φ respectively;
- (iii) The fiberwise operation $\Psi_x(\varphi|_{\mathcal{G}_x})$, $x \in M$.

For any $\Psi \in \Psi^\bullet(\mathcal{G})$, the vector representation of Ψ is the operator $\nu(\Psi) : C^\infty(M) \rightarrow C^\infty(M)$,

$$(\nu(\Psi)f) := \Psi_x(\mathbf{t}^* f)|_M.$$

Note that if $X \in \Gamma^\infty(A)$ is regarded as a differential operator on \mathcal{G} , then the vector representation of X is just $\nu(X)$, the image of X under the anchor map (regarded as a differential operator on M), so there is no confusion using the same notation for both.

2.2. Boundary groupoids. We define the main object we are interested in.

Definition 2.6. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid with M compact. We say that \mathcal{G} is a *boundary groupoid* if

- (i) The anchor map $\nu : A \rightarrow TM$ stratifies M into invariant sub-manifolds $M_0, M_1, \dots, M_r \subset M$;
- (ii) For all $k = 0, 1, \dots, r$, $\bar{M}_k := M_k \cup M_{k+1} \cup \dots \cup M_r$ are closed, immersed sub-manifolds of M ;
- (iii) $\mathcal{G}_0 := s^{-1}(M_0) = t^{-1}(M_0) \cong M_0 \times M_0$, the pair groupoid, and $\mathcal{G}_k := s^{-1}(M_k) = t^{-1}(M_k) \cong G_k \times (M_k \times M_k)$ for some Lie groups G_k ;
- (iv) For each k , there exists (unique) sub-bundles $\bar{A}_k \subset A|_{\bar{M}_k}$ such that $\bar{A}|_{M_k} = \ker(\nu|_{M_k})$.

For simplicity, we shall also assume that G_k and M_k are connected, hence all s -fibers are connected.

Notation 2.7. We shall also denote $\bar{\mathcal{G}}_k := \mathbf{s}^{-1}(\bar{M}_k) = \mathbf{t}^{-1}(\bar{M}_k)$.

Example 2.8. Let $M = M_0 \sqcup M_1$ be a manifold with embedded boundary [11]. The groupoid of space of totally characteristic operators is given (as a set) by

$$\mathcal{G} := (M_0 \times M_0) \sqcup \mathbb{R} \times (M_1 \times M_1).$$

Note that \mathcal{G} is an open dense subset of the blowup of M , known as the b-stretched product (see [12]).

Example 2.9. [17, Example 2.18] (See also [7]) Let $K = \mathrm{SU}(n)$, $T \subset K$ be the maximal torus, N be the Lie group of upper triangular matrices with unit diagonal.

Define the left action of T on $K \times N$ by

$$g \cdot (k, n) := (gk, gng^{-1}), \quad \forall (k, n) \in K \times N, g \in T.$$

It is easy to see that the projection onto $T \backslash (K \times N)$ is a submersion.

Define the groupoid operations on $\mathcal{G} := T \backslash (K \times N) \rightrightarrows T \backslash K$:

source and target maps: $\mathbf{s}(T(k, n)) = Tk, \mathbf{t}(T(k, n)) := Tk'$,

where $nk = k'a'n'$ is the (unique) Iwasawa decomposition;

multiplication: $\mathbf{m}(T(k_1, n_1), T(k_2, n_2)) := T(k_2, n_1n_2)$,

provided one has Iwasawa decomposition $n_2k_2 = k_1a'n'$;

inverse: $\mathbf{i}(T(k, n)) := T(k', n^{-1})$,

where $nk = k'a'n'$ is the (unique) Iwasawa decomposition;

unit: $\mathbf{u}(Tk) := T(k, e), e \in N$.

Remark that $T \backslash (K \times N) \rightrightarrows T \backslash K$ is just the symplectic groupoid of the Bruhat Poisson structure on K . In particular, when $n = 2$, $T \backslash K$ is just the sphere \mathbb{S}^2 . Let (x, y) be the stereographic coordinate opposite to T_e , the Poisson bi-vector field is

$$\Pi = (x^2 + y^2)(1 + x^2 + y^2)\partial_x \wedge \partial_y.$$

The Lie algebroid is $A = T^*\mathbb{S}^2$ and the anchor map is contraction with Π .

Example 2.10. Recall that any Poisson structure $\Pi \in \Gamma^\infty(\wedge^2 TM)$ defines a Lie algebroid structure on the cotangent bundle T^*M , and any bi-vector field Π on a two dimensional manifold M is Poisson (see, for example, [18]). In particular, let Π be any bi-vector field on the sphere \mathbb{S}^2 such that Π vanishes at exactly one point x_0 . Then the Lie algebroid is integrable [4]. It is easy to see that the groupoid integrating $T^*\mathbb{S}^2$ must be a boundary groupoid. Since on some fixed local coordinates around x_0 one can take $\Pi = f(x, y)\partial_x \wedge \partial_y$ for arbitrary smooth function f vanishing at x_0 only, differential operators obtained this way in general cannot be reduced to the cases considered in [10].

2.3. The Fredholmness criterion of Lauter and Nistor. Let $\mathcal{G} \rightrightarrows M$ be a boundary groupoid. Fix a metric on A , and hence on each \mathbf{s} -fiber \mathcal{G}_x of \mathcal{G} , as in the last section.

Let Ψ be pseudo-differential operator on \mathcal{G} . By right invariance, it is clear that

Lemma 2.11. *For any $x \in M$, Ψ_x is a uniformly bounded pseudo-differential operator on the manifold with bounded geometry \mathcal{G}_x .*

Moreover, since $\mathcal{G}_{M_0} \cong M_0 \times M_0$, we define a Riemannian metric on $M_0 \subseteq M$ by taking the metric on \mathcal{G}_x for any $x \in M_0$. Now since M_0 is a manifold with bounded geometry, we shall consider the ‘natural’ Sobolev spaces $\mathcal{L}^2(M_0)$ and $\mathcal{W}^m(M_0)$ as defined in Equation (19) in the appendix.

Recall that any uniformly bounded pseudo-differential operator of order m on the manifolds with bounded geometry \mathcal{G}_x (and M_0) extends to a bounded linear map from $\mathcal{W}^m(\mathcal{G}_x, \mathbf{t}^{-1}E)$ to $\mathcal{L}^2(\mathcal{G}_x, \mathbf{t}^{-1}E)$. With these notations, the Fredholmness criterion of Lauter and Nistor can be stated as:

Proposition 2.12. [6, Theorem 9] *Let $\Psi \in \Psi^{[m]}(\mathcal{G}, E)$ be elliptic. Then $\nu(\Psi) : \mathcal{W}^m(M_0, E) \rightarrow \mathcal{L}^2(M_0, E)$ is Fredholm if and only if, for all $x \in M \setminus M_0$, $\Psi_x : \mathcal{W}^m(\mathcal{G}_x, \mathbf{t}^{-1}E) \rightarrow \mathcal{L}^2(\mathcal{G}_x, \mathbf{t}^{-1}E)$ are invertible.*

3. THE CALCULUS WITH BOUNDS

3.1. A structural theorem for Lie algebroids. Given a boundary groupoid $\mathcal{G} \rightrightarrows M$. Fix any Riemannian metric \bar{g} on M . For each $k \geq 1$, let $d(\cdot, \bar{M}_k)$ be the distance function defined by g .

For each $k \geq 0$, fix a function $\rho_k \in C^\infty(M)$ such that $\rho_k > 0$ on $M \setminus \bar{M}_k$ and $\rho_k = d(\cdot, \bar{M}_k)$ on some open set containing \bar{M}_k .

Fix a metric g_A on A (i.e. a positive bi-linear form on A). Then g_A induces Riemannian metrics on the \mathbf{s} -fibers $\mathcal{G}_x := \mathbf{s}^{-1}(x)$ for each $x \in M$ by right invariance.

Consider the bundle map $d\rho_k \circ \nu : A \rightarrow \mathbb{R} \times M$.

Lemma 3.1. *For each k , there exists a constant ω_k such that for any x lying in some open neighborhood of \bar{M}_k , $X \in A_x$,*

$$(2) \quad |d\rho_k \circ \nu(X)| \leq \omega_k \rho_k(x) |X|_{g_A}.$$

Proof. Let $x_0 \in \bar{M}_k$ be arbitrary. Fix a coordinate chart $(\bar{U}, \bar{\mathbf{x}})$ around x . We may assume that $T\bar{M}_k^\perp \subset TM$ is trivial on \bar{U} . Then the map

$$(x_1 \cdots x_n) \mapsto \exp_{\bar{\mathbf{x}}(x_1, \dots, x_m)}(x_{m+1}, \dots, x_n),$$

where \exp is the exponential map defined by the Riemannian metric \bar{g} , defines a set of local coordinates on some open subset $U \subset M$. Moreover, by definition of the exponential map, one has

$$d\rho_k(\partial_i) = 0, \quad \forall i \leq m.$$

On the other hand, we may assume that A is trivial on U and fix any orthonormal basic sections $E_1, \dots, E_n \in \Gamma^\infty(A)$ and write

$$(3) \quad \nu(E_j) = \sum_{i=1}^n \nu_{ij} \partial_i$$

for some smooth functions ν_{ij} . Compositing with $d\rho_k$, one gets

$$d\rho_k \circ \nu(E_j) = \sum_{i=m+1}^n \nu_{ij}(\partial_i \rho_k).$$

Since the image of $A|_{\bar{M}_k}$ under ν lies in $T\bar{M}_k$, it follows that

$$\nu_{ij}(x) = 0, \quad \forall i > m \text{ and } x \in \bar{U} \subset \bar{M}_k.$$

The smoothness of ν_{ij} implies there exists an open subset $U' \subset U$ containing x_0 , and constant $\omega_{k,U}$ such that

$$|\nu_{ij}(x)| \leq \omega_{k,U} \rho_k(x), \quad \forall x \in U'.$$

Since x_0 is arbitrary, the lemma follows by considering a suitable finite cover of \bar{M}_k . \square

Remark 3.2. Given ω_k as in Lemma 3.1, we may modifying ρ_k outside a neighborhood of M_k to get

$$|d\rho_k \circ \nu(X)| \leq \omega_k \rho_k(x) |X|_{g_A}$$

Since we shall only be interested in estimates up to some multiples, it is clear that such modification have no effect on the arguments. Therefore we shall often implicitly assume such modification is being made if necessary.

Theorem 3.3. *For each k , let ω_k be defined in the previous Lemma 3.1. Suppose further that $|d\rho_k \circ \nu(X)| \leq \omega_k \rho_k(x) |X|_{g_A}$ for any $X \in A$. Then for any $x \in M, a, b \in \mathcal{G}_x$,*

$$(4) \quad \omega_k d(a, b) \geq \left| \log \left(\frac{\rho_k(\mathbf{t}(b))}{\rho_k(\mathbf{t}(a))} \right) \right|.$$

Proof. Given any $a, b \in \mathcal{G}_x$, since \mathcal{G}_x is a complete, connected Riemannian manifold, there exists a minimizing geodesic $\gamma : [0, 1] \rightarrow \mathcal{G}_x$ connecting a and b . Define the curves $\gamma_A(t) := dR_{(\gamma(t))^{-1}}(\gamma'(t)) \in A$, and $\gamma_M(t) := \mathbf{t} \circ \gamma(t) \in M$. Then one has the relation

$$\nu(\gamma_A(t)) = \gamma'_M(t).$$

Moreover, by right invariance, it follows that

$$d(a, b) = \text{length of } \gamma = \int_0^1 |\gamma_A(t)|_{g_A} dt.$$

Applying Lemma 3.1, one gets

$$\begin{aligned} \omega_k d(a, b) &= \int_0^1 C |\gamma_A(t)|_{g_A} dt \\ &\geq \int_0^1 \frac{|d\rho_k(\nu(\gamma_A(t)))|}{\rho_k(\gamma_M(t))} dt \geq \left| \int_0^1 \frac{d\rho_k(\gamma'_M(t))}{\rho_k(\gamma_M(t))} dt \right| = \left| \log \left(\frac{\rho_k(\mathbf{t}(b))}{\rho_k(\mathbf{t}(a))} \right) \right|. \end{aligned}$$

\square

Remark 3.4. Here, we observe that one can instead take any non-negative functions $\tilde{\rho}_k \in C^0(M)$, such that

- (i) $\tilde{\rho}_k = 0$ on \bar{M}_k , $\tilde{\rho}_k$ smooth and positive on $M \setminus M_k$;
- (ii) Theorem 3.3 holds for $\tilde{\rho}_k$ for some $\omega_k > 0$;
- (iii) One has $M\rho_k^\lambda \leq \tilde{\rho}_k \leq M'\rho_k^{\lambda'}$ for some $M, M', \lambda, \lambda' > 0$,

and all the subsequent arguments remain true. At this point it is unclear if there is an ‘optimal’ choice for the defining functions ρ_k .

Inspired by Lemma 3.1 and Theorem 3.3, we define:

Definition 3.5. The groupoid \mathcal{G} is said to be uniformly non-degenerate if there exist constants $\omega'_1, \omega'_2, \dots, \omega'_r > 0$ such that

$$|d\rho_k \circ \nu(X)| \geq \omega'_k \rho_k(x) |X|_{g_A},$$

for any $x \in M, X \in A_x$; The groupoid \mathcal{G} is said to be uniformly degenerate if there exist constants $\omega_1, \omega_2, \dots, \omega_r > 0$ and exponent $\lambda \geq 2$ such that

$$|d\rho_k \circ \nu(X)| \leq \omega_k (\rho_k(x))^\lambda |X|_{g_A},$$

for any $x \in M, X \in A_x$.

Remark 3.6. The groupoid \mathcal{G} is uniformly degenerate if and only if

$$d\nu_{ij}(x) = 0, \quad \forall i \leq m \text{ and } x \in \bar{U} \supset \bar{M}_k,$$

where ν_{ij} is defined in Equation (2).

Remark 3.7. If \mathcal{G} is uniformly non-degenerate, it is necessary that $\dim M_k = \dim M - k$.

Applying the same arguments as in Lemma 3.1 and Theorem 3.3, it is obvious that:

Corollary 3.8. *If \mathcal{G} is totally degenerate, then for any $\omega_k > 0$, there exists a function ρ'_k such that $\rho'_k = \rho_k$ on some open neighborhood of M_k (which depends on ω_k), and satisfies*

$$\omega_k d(a, b) \geq \left| \log \left(\frac{\rho'_k(\mathbf{t}(b))}{\rho'_k(\mathbf{t}(a))} \right) \right|,$$

for all $a, b \in \mathcal{G}$ such that $\mathbf{s}(a) = \mathbf{s}(b)$.

3.2. Construction of the calculus with bounds. Given a Hausdorff groupoid \mathcal{G} , defined the manifold $\tilde{\mathcal{G}} := \{(a, b) \in \mathcal{G} \times \mathcal{G} : \mathbf{s}(a) = \mathbf{s}(b)\}$. Let $\tilde{\mathbf{m}}$ denote the map

$$(a, b) \mapsto ab^{-1}, \quad (a, b) \in \tilde{\mathcal{G}}.$$

Also recall that for any $X \in \Gamma^\infty(A)$, X determines a right invariant vector field $X^{\mathbf{r}} \in \Gamma^\infty(\text{Ker}(ds))$. Moreover, given vector fields on $X^{\mathbf{r}}, Y^{\mathbf{r}}$ on \mathcal{G} ,

$$X^{\mathbf{r}}(a) \oplus Y^{\mathbf{r}}(b) \in T_{(a,b)} \tilde{\mathcal{G}} \subset T_{(a,b)}(\mathcal{G} \times \mathcal{G})$$

for any $(a, b) \in \tilde{\mathcal{G}}$. We shall consider $X^{\mathbf{r}} \oplus Y^{\mathbf{r}}$ as a vector field on $\tilde{\mathcal{G}}$.

Consider $d\tilde{\mathbf{m}}(X^{\mathbf{r}} \oplus Y^{\mathbf{r}})$. Observe that $X^{\mathbf{r}}$ is just the vector field

$$X^{\mathbf{r}}(a) = \partial_t|_{t=0} \exp tX(\mathbf{t}(a))(a), \quad \forall a \in \mathcal{G}.$$

It follows that for any $X, Y \in \Gamma^\infty(A), (a, b) \in \tilde{\mathcal{G}}$,

$$\begin{aligned} (5) \quad d\tilde{\mathbf{m}}(X^{\mathbf{r}} \oplus Y^{\mathbf{r}})(ab^{-1}) &= \partial_t|_{t=0} (\exp tX(\mathbf{t}(a))) ab^{-1} (\exp tY(\mathbf{t}(b)))^{-1} \\ &= \partial_t|_{t=0} (\exp tX(\mathbf{t}(c))) c (\exp tY(\mathbf{t}(c)))^{-1}, \end{aligned}$$

where $c := ab^{-1}$. In particular, $d\tilde{\mathbf{m}}(X^{\mathbf{r}} \oplus Y^{\mathbf{r}})$ is a well defined vector field on \mathcal{G} .

Recall how the \mathbf{s} -fiberwise covariant derivatives of a function is defined. Let ∇ be the Levi-Cevita A -connection with respect to the given Riemannian metric g_A . By right invariance, ∇ defines the Levi-Cevita connection on each \mathbf{s} -fiber \mathcal{G}_x , which

we still denote by ∇ . For any smooth functions ψ on \mathcal{G} , $\nabla^l \psi \in \Gamma^\infty(\otimes^l \ker(ds))$, $l = 1, 2, \dots$, is defined inductively by

$$(6) \quad \begin{aligned} \nabla \psi(X_1^{\mathbf{r}}) &:= L_{X_1^{\mathbf{r}}} \psi \\ \nabla^l \psi(X_1^{\mathbf{r}}, X_2^{\mathbf{r}}, \dots, X_l^{\mathbf{r}}) &:= L_{X_1^{\mathbf{r}}}(\nabla^{l-1} \psi(X_2^{\mathbf{r}}, \dots, X_l^{\mathbf{r}})) \\ &\quad - \sum_{k=2}^l \nabla^{l-1} \psi(X_2^{\mathbf{r}}, \dots, (\nabla_{X_1} X_k)^{\mathbf{r}}, \dots, X_l^{\mathbf{r}}). \end{aligned}$$

Likewise, on $\tilde{\mathcal{G}}$, let $\tilde{\nabla}$ be the Cartesian product connection. One considers higher covariant derivatives $\tilde{\nabla}^l$. In particular, observe that for any $\psi \in C^\infty(\mathcal{G})$,

$$(7) \quad \begin{aligned} \tilde{\nabla}(\tilde{\mathbf{m}}^* \psi)(V_1^{\mathbf{r}} \oplus W_1^{\mathbf{r}})(a, b) &= L_{d\tilde{\mathbf{m}}(V_1^{\mathbf{r}} \oplus W_1^{\mathbf{r}})}(ab^{-1}) \\ \tilde{\nabla}^2(\tilde{\mathbf{m}}^* \psi)(V_1^{\mathbf{r}} \oplus W_1^{\mathbf{r}}, V_2^{\mathbf{r}} \oplus W_2^{\mathbf{r}})(a, b) &= L_{d\tilde{\mathbf{m}}(V_1^{\mathbf{r}} \oplus W_1^{\mathbf{r}})} L_{d\tilde{\mathbf{m}}(V_2^{\mathbf{r}} \oplus W_2^{\mathbf{r}})} \psi(ab^{-1}) \\ &\quad - L_{d\tilde{\mathbf{m}}(\nabla_{V_1} V_2^{\mathbf{r}} \oplus \nabla_{W_1} W_2^{\mathbf{r}})} \psi(ab^{-1}), \end{aligned}$$

and so on.

For each $(a, b) \in \tilde{\mathcal{G}}$, define $d(a, b)$ to be the Riemannian distance on $\mathcal{G}_{\mathbf{s}(a)} = \mathcal{G}_{\mathbf{s}(b)}$ between a and b .

Definition 3.9. For each $\varepsilon > 0$, define the *exponentially decaying calculus of order ε* to be the space of kernels

$$\begin{aligned} \Psi_{\varepsilon; \mathbf{0}}^{-\infty}(\mathcal{G}) &:= \left\{ \psi \in C^0(\mathcal{G}) : \psi|_{\mathcal{G}_k} \in C^\infty(\mathcal{G}_k), \forall k \in \mathbb{N}, \exists M_l > 0 \text{ such that} \right. \\ &\quad \text{for any } X_1, \dots, X_l, Y_1, \dots, Y_l \in \Gamma^\infty(\mathbf{A}), (a, b) \in \tilde{\mathcal{G}}, \\ &\quad \tilde{\nabla}^l(\mathbf{m}^* \psi)(X_1^{\mathbf{r}} \oplus Y_1^{\mathbf{r}}, \dots, X_l^{\mathbf{r}} \oplus Y_l^{\mathbf{r}})(a, b) \\ &\quad \left. \leq M_l e^{-\varepsilon' d(a, b)} (|X_1| + |Y_1|)(|X_2| + |Y_2|) \cdots (|X_l| + |Y_l|) \right\}. \end{aligned}$$

Remark 3.10. For simplicity we only consider the scalar case. A groupoid pseudo-differential operators on a vector bundle $E \rightarrow M$ can be identified with a (distributional) section on $\mathbf{t}^{-1}E \otimes \mathbf{s}^{-1}E \rightarrow \mathcal{G}$. One instead considers covariant derivative on E and it is clear that all arguments below follows.

As in the case of manifolds with boundary [10, 11], we compute the composition rule of the calculus.

Lemma 3.11. *For any $\varepsilon_1, \varepsilon_2 \geq 0$*

$$\Psi_{\varepsilon_1; \mathbf{0}}^{-\infty} \circ \Psi_{\varepsilon_2; \mathbf{0}}^{-\infty} \subseteq \Psi_{\min\{\varepsilon_1, \varepsilon_2\}}^{-\infty}.$$

Proof. For simplicity we only consider the scalar case. It suffices to consider the convolution product $\psi_1 \circ \psi_2$ for any $\psi_1 \in \Psi_{\varepsilon_1; \mathbf{0}}^{-\infty}(\mathcal{G})$, $\psi_2 \in \Psi_{\varepsilon_2; \mathbf{0}}^{-\infty}(\mathcal{G})$. In view of the formula

$$\psi_1 \circ \psi_2(a) = \int_{b \in \mathcal{G}_{\mathbf{s}(a)}} \psi_1(ab^{-1}) \psi_2(b) \mu_{\mathbf{s}(a)}(b) = \int_{c \in \mathbf{s}^{-1}(\mathbf{t}(a))} \psi_1(c^{-1}) \psi_2(ca) \mu_{\mathbf{t}(a)}(c),$$

one can without loss of generality assume $\varepsilon_1 \leq \varepsilon_2$. Then by definition one has the estimates $\psi_1(a) \leq M e^{-\varepsilon_1' d(a, \mathbf{s}(a))}$, $\psi_2(a) \leq M' e^{-\varepsilon_2' d(a, \mathbf{s}(a))}$ for some $\varepsilon_1' > \varepsilon_1$, $\varepsilon_2' > \varepsilon_2$ and constants $M, M' > 0$. One may further assume that $\varepsilon_1' < \varepsilon_2'$.

The hypothesis implies for any $a \in \mathcal{G}$

$$\begin{aligned}
|\psi_1 \circ \psi_2(a)| &\leq MM' \int_{b \in \mathcal{G}_{\mathbf{s}(a)}} e^{-\varepsilon'_1 d(a,b)} e^{-\varepsilon'_2 d(b, \mathbf{s}(b))} \mu_{\mathbf{s}(a)}(b) \\
&\leq MM' \int_{b \in \mathcal{G}_{\mathbf{s}(a)}} e^{-\varepsilon'_1 |d(a, \mathbf{s}(a)) - d(b, \mathbf{s}(b))| - \varepsilon'_2 d(b, \mathbf{s}(b))} \mu_{\mathbf{s}(a)}(b) \\
&= MM' \int_{b \in B_a} e^{-\varepsilon'_1 d(a, \mathbf{s}(a))} e^{-(\varepsilon'_2 - \varepsilon'_1) d(b, \mathbf{s}(b))} \mu_{\mathbf{s}(a)}(b) \\
&\quad + MM' \int_{b \notin B_a} e^{\varepsilon'_1 d(a, \mathbf{s}(a))} e^{-(\varepsilon'_2 + \varepsilon'_1) d(b, \mathbf{s}(b))} \mu_{\mathbf{s}(a)}(b),
\end{aligned}$$

where B_a denotes the set $\{b \in \mathcal{G}_{\mathbf{s}(a)} : d(b, \mathbf{s}(b)) < d(a, \mathbf{s}(a))\}$ for each a . Hence for the first integral, one has

$$\begin{aligned}
&\int_{b \in B_a} e^{-\varepsilon'_1 d(a, \mathbf{s}(a))} e^{-(\varepsilon'_2 - \varepsilon'_1) d(b, \mathbf{s}(b))} \mu_{\mathbf{s}(a)}(b) e^{-\varepsilon'_1 d(a, \mathbf{s}(a))} \\
&= e^{-\varepsilon'_1 d(a, \mathbf{s}(a))} \int_{b \in B_a} e^{-(\varepsilon'_2 - \varepsilon'_1) d(b, \mathbf{s}(b))} \mu_{\mathbf{s}(a)}(b),
\end{aligned}$$

which is finite and only depends on $\mathbf{s}(a)$. As for the second integral, write

$$\begin{aligned}
&\varepsilon'_1 d(a, \mathbf{s}(a)) - (\varepsilon'_2 + \varepsilon'_1) d(b, \mathbf{s}(b)) \\
&= -\varepsilon'_1 d(a, \mathbf{s}(a)) + 2\varepsilon'_1 (d(a, \mathbf{s}(a)) - d(b, \mathbf{s}(b))) - (\varepsilon'_2 - \varepsilon'_1) d(b, \mathbf{s}(b)).
\end{aligned}$$

Since $d(b, \mathbf{s}(b)) \geq d(a, \mathbf{s}(a))$ for any $b \notin B_a$. It follows that the second integral is again bounded by

$$e^{-\varepsilon'_1 d(a, \mathbf{s}(a))} \int_{b \in \mathcal{G}_{\mathbf{s}(a)}} e^{-(\varepsilon'_2 - \varepsilon'_1) d(b, \mathbf{s}(b))} \mu_{\mathbf{s}(a)}(b).$$

Adding the two together and rearranging, one gets $e^{\varepsilon'_1 d_{\mathbf{s}}(a)}(\psi_1 \circ \psi_2)(a)$ is a bounded function, as asserted.

To prove the assertion for derivatives, observe that by right invariance of μ ,

$$(\psi_1 \circ \psi_2)(ab^{-1}) = \int \psi_1(ac^{-1}) \psi_2(cb^{-1}) \mu_{\mathbf{s}(a)}(c),$$

for any $(a, b) \in \tilde{\mathcal{G}}$. It follows that for any $(a, b) \in \tilde{\mathcal{G}}$, $X, Y \in \Gamma^\infty(\mathbf{A})$,

$$L_{d\tilde{\mathbf{m}}(X^{\mathbf{r}} \oplus Y^{\mathbf{r}})}(\psi_1 \circ \psi_2)(ab^{-1}) = \int (L_{d\tilde{\mathbf{m}}(X^{\mathbf{r}} \oplus 0)} \psi_1)(ac^{-1}) (L_{d\tilde{\mathbf{m}}(0 \oplus Y^{\mathbf{r}})} \psi_2)(cb^{-1}) \mu_{\mathbf{s}(a)}(c).$$

and so on for higher derivatives. \square

Next, we write down the definition of the calculus with bounds. For each k , denote

$$\hat{\rho}_k := ((\mathbf{s}^* \rho_k)^2 + (\mathbf{t}^* \rho_k)^2)^{\frac{1}{2}} \in C^0(\mathcal{G}).$$

Definition 3.12. Let \mathcal{G} be uniformly degenerate. For each $\varepsilon > 0, \lambda_1, \dots, \lambda_r \geq 0$, the calculus with bounds of order $-\infty$ is defined to be

$$\begin{aligned} \Psi_{\varepsilon_1; \lambda_1, \dots, \lambda_r}^{-\infty}(\mathcal{G}) &:= \left\{ \psi \in C^0(\mathcal{G}) : \psi|_{\mathcal{G}_k} \in C^\infty(\mathcal{G}_k), \forall l \in \mathbb{N}, \exists M_l > 0 \text{ such that} \right. \\ &\quad \text{for any } X_1, \dots, X_l, Y_1, \dots, Y_l \in \Gamma^\infty(\mathbf{A}), (a, b) \in \tilde{\mathcal{G}} \\ &\quad \tilde{\nabla}^l(\tilde{\mathbf{m}}^* \psi)(X_l^{\mathbf{r}} \oplus Y_l^{\mathbf{r}}, \dots, X_1^{\mathbf{r}} \oplus Y_1^{\mathbf{r}})(a, b) \\ &\quad \left. \leq M_l e^{-\varepsilon' d(a, b)} \prod_{i=1}^l (|X_i| + |Y_i|) \prod_{i=1}^l \hat{\rho}_i^{\lambda_i} \right\}. \end{aligned}$$

With the new filtration we need to refine the composition rule.

Theorem 3.13. *Given any collection of data $\varepsilon_1, \varepsilon_2 > 0, \lambda_1^{(1)}, \dots, \lambda_r^{(1)}, \lambda_1^{(2)}, \dots, \lambda_r^{(2)} \geq 0$. Suppose that*

$$\varepsilon_3 := \min \left\{ \varepsilon_1 - \sum_{k=1}^r \omega_k \lambda_k^{(2)}, \varepsilon_2 - \sum_{k=1}^r \omega_k \lambda_k^{(1)} \right\} > 0,$$

with ω_k is as in Theorem 3.3. Then the convolution product between any pair of elements in $\Psi_{\varepsilon_1; \lambda_1^{(1)}, \dots, \lambda_r^{(1)}}^{-\infty}(\mathcal{G})$ and $\Psi_{\varepsilon_2; \lambda_1^{(2)}, \dots, \lambda_r^{(2)}}^{-\infty}(\mathcal{G})$ is well defined. Moreover, one has

$$(8) \quad \Psi_{\varepsilon_1; \lambda_1^{(1)}, \dots, \lambda_r^{(1)}}^{-\infty}(\mathcal{G}) \circ \Psi_{\varepsilon_2; \lambda_1^{(2)}, \dots, \lambda_r^{(2)}}^{-\infty}(\mathcal{G}) \subseteq \Psi_{\varepsilon_3; \lambda_1^{(1)} + \lambda_1^{(2)}, \dots, \lambda_r^{(1)} + \lambda_r^{(2)}}^{-\infty}(\mathcal{G}).$$

Proof. For any $\psi_1 \in \Psi_{\varepsilon_1; \lambda_1^{(1)}, \dots, \lambda_r^{(1)}}^{-\infty}(\mathcal{G})$, $\psi_2 \in \Psi_{\varepsilon_2; \lambda_1^{(2)}, \dots, \lambda_r^{(2)}}^{-\infty}(\mathcal{G})$, the convolution product reads (provided the integral is finite):

$$\begin{aligned} \psi_1 \circ \psi_2(a) &= \int_{b \in \mathcal{G}_{\mathbf{s}(a)}} u_1(ab^{-1}) u_2(b) \mu_{\mathbf{s}(a)}(b) \\ &\leq M_1 \int_{b \in \mathcal{G}_{\mathbf{s}(a)}} e^{-\varepsilon'_1 d(a, b)} e^{-\varepsilon'_2 d(b, \mathbf{s}(b))} \\ &\quad \times \prod_{i=1}^r ((\mathbf{s}^* \rho_i(ab^{-1}))^2 + (\mathbf{t}^* \rho_i(ab^{-1}))^2)^{\frac{\lambda_i^{(1)}}{2}} ((\mathbf{s}^* \rho_i(b))^2 + (\mathbf{t}^* \rho_i(b))^2)^{\frac{\lambda_i^{(2)}}{2}} \mu_{\mathbf{s}(a)}(b). \end{aligned}$$

For some $\varepsilon'_1 > \varepsilon_1, \varepsilon'_2 > \varepsilon_2$. Consider the product term in the integrand, one has

$$\begin{aligned} \mathbf{s}^* \rho_i(ab^{-1}) &= \rho_i(\mathbf{t}(b)) \leq e^{\omega_i d(b, \mathbf{s}(b))} \rho_i(\mathbf{s}(a)) \\ \mathbf{t}^* \rho_i(ab^{-1}) &= \rho_i(\mathbf{t}(a)) \\ \mathbf{t}^* \rho_i(b) &= \rho_i(\mathbf{t}(b)) \leq e^{\omega_i d(b, a)} \rho_i(\mathbf{t}(a)) \\ \mathbf{s}^* \rho_i(b) &= \rho_i(\mathbf{s}(a)), \end{aligned}$$

where we used Theorem 3.3 for the first and third line. Hence, one estimates the integrand

$$\begin{aligned} e^{-\varepsilon'_1 d(a, b)} e^{-\varepsilon'_2 d(b, \mathbf{s}(b))} \prod_{i=1}^r ((\mathbf{s}^* \rho_i(ab^{-1}))^2 + (\mathbf{t}^* \rho_i(ab^{-1}))^2)^{\frac{\lambda_i^{(1)}}{2}} ((\mathbf{s}^* \rho_i(b))^2 + (\mathbf{t}^* \rho_i(b))^2)^{\frac{\lambda_i^{(2)}}{2}} \\ \leq \left(\prod_{i=1}^r \hat{\rho}_i(a)^{\lambda_i^{(1)} + \lambda_i^{(2)}} \right) e^{-(\varepsilon'_1 - \sum_{i=1}^r \omega_i \lambda_i^{(2)}) d(a, b)} e^{-(\varepsilon'_2 - \sum_{i=1}^r \omega_i \lambda_i^{(1)}) d(b, \mathbf{s}(b))}. \end{aligned}$$

The theorem follows by factoring out the term $\prod_{i=1}^r \hat{\rho}_k(a)^{\lambda_k^{(1)} + \lambda_k^{(2)}}$, and then following the arguments of Lemma 3.11. \square

In the case \mathcal{G} is uniformly degenerate, since ω_k can be made arbitrary small, it follows that convolution is always defined and

$$(9) \quad \Psi_{\varepsilon_1; \lambda_1^{(1)}, \dots, \lambda_r^{(1)}}^{-\infty}(\mathcal{G}) \circ \Psi_{\varepsilon_2; \lambda_1^{(2)}, \dots, \lambda_r^{(2)}}^{-\infty}(\mathcal{G}) \subseteq \Psi_{\min\{\varepsilon_1, \varepsilon_2\}; \lambda_1^{(1)} + \lambda_1^{(2)}, \dots, \lambda_r^{(1)} + \lambda_r^{(2)}}^{-\infty}(\mathcal{G}).$$

We turn to study convolution of singular kernels.

Lemma 3.14. *For any $m \in \mathbb{Z}, \varepsilon > 0, \lambda_1, \dots, \lambda_r \geq 0$,*

$$(10) \quad \Psi^{[m]}(\mathcal{G}) \circ \Psi_{\varepsilon_1; \lambda_1, \dots, \lambda_r}^{-\infty}(\mathcal{G}) \subset \Psi_{\varepsilon_1; \lambda_1, \dots, \lambda_r}^{-\infty}(\mathcal{G}).$$

Proof. Suppose we are given kernels $\psi \in \Psi^{[m]}(\mathcal{G}), \kappa \in \Psi_{\varepsilon_1; \lambda_1, \dots, \lambda_r}^{-\infty}(\mathcal{G})$. Fix any elliptic differential operator $\Delta \in \Psi^{[m']}(\mathcal{G})$ with order $m' > m + \dim \mathcal{G} - \dim M$. Let $Q \in \Psi^{[-m']}(\mathcal{G})$ be a parametrix of Δ . In other words, one has

$$S := \text{id} - Q\Delta \in \Psi^{-\infty}(\mathcal{G}).$$

Then one can write

$$\psi \circ \kappa = (\psi S) \circ \kappa + (\psi Q) \circ (\Delta \kappa).$$

By definition, 3.12, $\kappa \in \Psi_{\varepsilon_1; \lambda_1, \dots, \lambda_r}^{-\infty}(\mathcal{G})$ implies $\Delta \kappa$ lies in the same space.

On the other hand, ψQ is a uniformly supported pseudo-differential operator of order less than $(-\dim \mathcal{G} + \dim M)$. Therefore it is a classical result (see [5]) that ψQ is continuous kernel on \mathcal{G} with compact support. It follows that the proof of 3.13 applies and $(\psi Q) \circ (\Delta \kappa) \in \Psi_{\varepsilon; \infty}^{-\infty}(\mathcal{G})$. The same argument holds for $(\psi S) \circ \kappa$. Hence we conclude that

$$\psi \circ \kappa \in \Psi_{\varepsilon_1; \lambda_1, \dots, \lambda_r}^{-\infty}(\mathcal{G}),$$

as asserted. \square

By considering adjoint of Lemma 3.14, it is obvious that

$$(11) \quad \Psi_{\varepsilon_1; \lambda_1, \dots, \lambda_r}^{-\infty}(\mathcal{G}) \circ \Psi^{[m]} \subset \Psi_{\varepsilon_1; \lambda_1, \dots, \lambda_r}^{-\infty}(\mathcal{G}),$$

as well.

Notation 3.15. For $1 \leq k \leq r-1$, denote

$$(12) \quad \Psi_{\varepsilon; \lambda_1, \dots, \lambda_k, \infty}^{-\infty}(\mathcal{G}) := \bigcap_{\lambda_{k+1} > 0} \Psi_{\varepsilon; \lambda_1, \dots, \lambda_k, \lambda_{k+1}, \dots, \lambda_r}^{-\infty}(\mathcal{G}).$$

Note that $\Psi_{\varepsilon; \lambda_1, \dots, \lambda_k, \infty}^{-\infty}(\mathcal{G})$ is uniquely defined since $\rho_1 \leq \rho_2 \leq \dots \leq \rho_r$.

If for some $\lambda_1 = \dots = \lambda_{r'} = 0$ for some $r' \leq r$, we shall denote $\Psi_{\varepsilon; \lambda_1, \dots, \lambda_r}^{-\infty}(\mathcal{G})$ by

$$\Psi_{\varepsilon; \mathbf{0}_{r'}, \lambda_{r'+1}, \dots, \lambda_r}^{-\infty}(\mathcal{G}).$$

For convenience, we shall also write

$$\Psi_{\bullet, \mathbf{0}_{r'}, \lambda_{r'+1}, \dots, \lambda_r}^{-\infty}(\mathcal{G}) := \bigcup_{\varepsilon > 0} \Psi_{\varepsilon; \mathbf{0}_{r'}, \lambda_{r'+1}, \dots, \lambda_r}^{-\infty}(\mathcal{G}).$$

Lemma 3.16. *For any $\psi \in \Psi_{\varepsilon; \lambda_1, \dots, \lambda_k, \infty}^{-\infty}(\mathcal{G})$, all derivatives of ψ vanish at $\bar{\mathcal{G}}_{k+1}$ and $\psi|_{\mathcal{G}_k}$ is smooth.*

Proof. It suffices to consider the derivatives on some coordinates patches. For any $a \in \bar{\mathcal{G}}_{k+1}$, let

$$\mathbf{x}_{Z_I}^{(\alpha)}(\tau, x) := \exp(\tau \cdot (X_1^{(\alpha)}, \dots, X_k^{(\alpha)})) \exp Z_I(x)$$

be an exponential coordinates patch defined in Definition B.4 around a . We may assume that there exist constant $C > 0$ such that $d(b, \mathbf{s}(b)) < C$ on $U_{Z_I}^{(\alpha)}$. Then

$$\hat{\rho}_{k+1}^2(\mathbf{x}_{Z_I}^{(\alpha)}(\tau, x)) \leq \rho_{k+1}^2(\mathbf{x}_{Z_I}^{(\alpha)}(\tau, x)) + e^{C\omega} \rho_{k+1}^2(\mathbf{x}_{Z_I}^{(\alpha)}(\tau, x)),$$

for some $\omega > 0$. Therefore the assumption $\psi \in \Psi_{\varepsilon; \lambda_1, \dots, \lambda_k, \infty}^{-\infty}(\mathcal{G})$ implies

$$\psi \leq (1 + e^{C\omega})(x_{\dim \mathcal{G}_{k+1}+1}^2 + \dots + x_{\dim \mathcal{G}}^2)^N$$

for any $N > 0$, which in turn implies all derivatives of ψ at the subset $\{x_1 = \dots = x_{\dim \mathcal{G}_{k+1}}\}$ vanishes.

Since by definition, ψ is smooth on \mathcal{G}_k and $\bar{\mathcal{G}}_k = \mathcal{G}_k \cup \bar{\mathcal{G}}_{k+1}$, it follows that ψ is smooth on $\bar{\mathcal{G}}_k$. \square

Definition 3.17. Given r sequences of number $\{\lambda_i^{(1)}\}, \dots, \{\lambda_i^{(r)}\}$, such that for all i , $\lambda_1^{(k)} = 0$, and $\{\lambda_i^{(k)}\} \rightarrow \infty$ as $i \rightarrow \infty$. Given r sequences of kernels $\{\psi_i^{(k)}\}, k = 1, \dots, r$, such that

$$\psi_i^{(k)} \in \Psi_{\varepsilon; \mathbf{0}_{k-1}, \lambda_i^{(k)}, \infty}^{-\infty}(\mathcal{G}),$$

and furthermore satisfy the smoothness conditions

$$\psi_1^{(k)} \in C^\infty(\mathcal{G}), \quad \psi_i^{(k)}|_{\mathcal{G} \setminus \bar{\mathcal{G}}_k} \in C^\infty(\mathcal{G} \setminus \bar{\mathcal{G}}_k), \text{ for all } i \geq 2.$$

We say that an element $\psi \in \Psi_{\varepsilon; \mathbf{0}}^{-\infty}(\mathcal{G})$ has an asymptotic expansion

$$\psi \sim \sum_{k=1}^r \left(\sum_{i=1}^{\infty} \psi_i^{(k)} \right),$$

if for any sets of indexes i_1, \dots, i_r ,

$$\psi - \sum_{k=1}^r \left(\sum_{i=1}^{i_k} \psi_i^{(k)} \right) \in \Psi_{\varepsilon; \lambda_{1+i_1}^{(1)}, \dots, \lambda_{1+i_r}^{(r)}}^{-\infty}(\mathcal{G}).$$

The space of kernels with asymptotic expansion is denoted by $\Psi_\varepsilon^{-\infty}(\mathcal{G})$, and we write $\Psi_\bullet^{-\infty}(\mathcal{G}) := \bigcap_{\varepsilon > 0} \Psi_\varepsilon^{-\infty}(\mathcal{G})$.

4. COMPACT PARAMETRIX AND GENERALIZED INVERSE OF FREDHOLM OPERATORS

4.1. Extension of exponentially decaying kernels. The following assumption is crucial in our construction of parametrices and inverses of uniformly supported pseudo-differential operator on \mathcal{G} .

Definition 4.1. Let \mathcal{G} be a boundary groupoid, not necessary uniformly degenerate. We say that \mathcal{G} satisfies the *extension property* if for any $k \leq r$, $\varphi \in \Psi_{\bullet; \infty}^{-\infty}(\bar{\mathcal{G}}_k)$, $\varphi_0 \in \Psi_{\bullet; \mathbf{0}_k}^{-\infty}(\mathcal{G})$, $\kappa \in \Psi^{-\infty}(\mathcal{G})$, and differential operator $D \in \Psi^{[m]}(\mathcal{G})$ satisfying the relation

$$D|_{\bar{\mathcal{G}}_k}(\varphi_0|_{\bar{\mathcal{G}}_k} + \varphi) - \kappa|_{\bar{\mathcal{G}}_k} = 0,$$

there exists an extension $\bar{\psi} \in \Psi_{\bullet; \mathbf{0}_k, \infty}^{-\infty}(\mathcal{G})$ such that

- (i) $\bar{\varphi}|_{\bar{\mathcal{G}}_k} = \varphi$;
- (ii) $D(\varphi_0 + \bar{\varphi}) - \kappa \in \Psi_{\bullet; \mathbf{0}_{k-1}, \lambda_k, \infty}^{-\infty}(\mathcal{G})$, for some $\lambda_k > 0$.

We say that \mathcal{G} has the smooth extension property if for any $\varphi \in \Psi_{\bullet; \infty}^{-\infty}(\bar{\mathcal{G}}_k)$, φ_0 with asymptotic expansion $\varphi_0 \sim \sum_{j=k+1}^r \sum_{i=1} \varphi_i^{(j)}$, $\varphi_i^{(j)} \in \Psi_{\varepsilon; \mathbf{0}_j, \lambda_i^{(j)}, \infty}^{-\infty}(\mathcal{G})$, and pseudo-differential operator $\Psi \in \Psi^{[m]}(\mathcal{G})$ satisfying the relation

$$\Psi|_{\bar{\mathcal{G}}_k}(\varphi_0|_{\bar{\mathcal{G}}_k} + \varphi) - \kappa|_{\bar{\mathcal{G}}_k} = 0,$$

there exists an extension $\bar{\varphi} \in \Psi_{\bullet; \mathbf{0}_k, \infty}^{-\infty}(\mathcal{G}) \cap C^\infty(\mathcal{G})$ such that $\kappa - \Psi(\varphi_0 + \bar{\varphi}) \in \Psi_{\bullet; \mathbf{0}_{k-1}, \lambda, \infty}^{-\infty}(\mathcal{G})$ for some $\lambda > 0$.

In the appendix (See Propositions B.8 and B.9), we shall prove that

Theorem 4.2. *Any boundary groupoid of the form*

$$\mathcal{G} = (\mathbf{M}_0 \times \mathbf{M}_0) \bigsqcup \mathbf{G} \times (\mathbf{M}_1 \times \mathbf{M}_1),$$

where \mathbf{G} is a nilpotent Lie group, satisfies the extension property.

The proof is elementary (but tedious). Indeed we conjecture that:

Conjecture 4.3. Every boundary groupoid \mathcal{G} with polynomial volume growth satisfies the smooth extension property.

In the following we shall assume that the groupoid \mathcal{G} satisfies the extension property.

4.2. Inverse and parametrix when $\mathcal{G} = \mathbf{M}_0 \times \mathbf{M}_0 \bigsqcup \mathbf{M}_1 \times \mathbf{M}_1 \times \mathbf{G}$. The main tool we use is the following estimate from Shubin:

Lemma 4.4. [16] *Let \mathbf{M} be any manifold with bounded geometry. Fix $r > 0$. For any invertible, uniformly supported pseudo-differential operators Ψ of order m on \mathbf{M} , Ψ^{-1} is pseudo-differential operator of order $-m$, where the distributional kernel φ of Ψ^{-1} satisfies the estimate*

$$(13) \quad \partial_x^I \partial_y^J \varphi(x, y) \leq M_{IJ} (1 + d(x, y)^{m-|I|-|J|-\dim \mathbf{M}}) e^{-\varepsilon d(x, y)},$$

for some $\varepsilon > 0$ and for all multi-indices I, J and all $(x, y) \in \mathbf{M} \times \mathbf{M} \setminus \mathbf{M}$. Moreover, if $m - \dim \mathbf{M} > 0$, then $\varphi \in C^j(\mathbf{M} \times \mathbf{M})$ for any $j < m - \dim \mathbf{M}$.

The first step of our construction is to describe compact parametrix of an elliptic operator.

Theorem 4.5. *Let $\mathcal{G} = \mathbf{M}_0 \times \mathbf{M}_0 \bigsqcup \mathbf{G} \times \mathbf{M}_1 \times \mathbf{M}_1$, Given any elliptic, groupoid differential operator $\Psi = \{\Psi\}_{x \in \mathbf{M}} \in \Psi^{[m]}(\mathcal{G})$. Suppose that for all $x \notin \mathbf{M}_0$, Ψ_x is invertible, then its vector representation $\nu(\Psi)$ is Fredholm. Moreover, there exists $\Phi \in \Psi^{-[m]}(\mathcal{G}) \oplus \Psi_{\bullet; \mathbf{0}}^{-\infty}(\mathcal{G})$ such that*

$$\text{id} - \nu(\Psi)\nu(\Phi) : \mathcal{L}^2(\mathbf{M}_0) \rightarrow \mathcal{L}^2(\mathbf{M}_0)$$

is compact.

Proof. The proof of the lemma closely follows the proof of [17, Theorem 3.21].

By standard arguments there exists $Q \in \Psi^{[-m]}(\mathcal{G})$ such that

$$\text{id} - \Psi Q, \text{id} - Q\Psi \in \Psi^{-\infty}(\mathcal{G}).$$

Observe that $(\Psi|_{\mathcal{G}_1})^{-1}$ is right invariant by uniqueness of the inverse operator. By Lemma 4.4, one has

$$(\Psi|_{\mathcal{G}_1})^{-1} \in \Psi^{[-m]}(\mathcal{G}_1) \oplus \Psi_{\varepsilon}^{-\infty}(\mathcal{G}_1),$$

for some $\varepsilon > 0$. It follows that

$$(\Psi|_{\mathcal{G}_1})^{-1} - Q|_{\mathcal{G}_1} \in \Psi_{\varepsilon}^{-\infty}(\mathcal{G}_1).$$

The extension property guarantees that there exists $S \in \Psi_{\varepsilon',0}^{-\infty}(\mathcal{G})$, for some $\varepsilon' > 0$, such that

$$S|_{\mathcal{G}_1} = (\Psi|_{\mathcal{G}_1})^{-1} - Q|_{\mathcal{G}_1}.$$

Define

$$(14) \quad \Phi := Q + S.$$

Then $\Phi|_{\mathcal{G}_1} = (\Psi|_{\mathcal{G}_1})^{-1}$. It follows that

$$(\text{id} - \Psi\Phi)|_{\mathcal{G}_1} = \text{id} - (\Psi|_{\mathcal{G}_1})(\Phi|_{\mathcal{G}_1}) = 0 = (\text{id} - \Phi\Psi)|_{\mathcal{G}_1}.$$

Therefore by [6], $\text{id} - \nu(\Psi)\nu(\Phi)$ is compact. \square

Assume further that \mathcal{G} is uniformly degenerate. Then one can improve the parametrix by considering the Neumann series, as in [10].

Lemma 4.6. *Let $\Psi \in \Psi^{[m]}(\mathcal{G})$ be as in Theorem 4.5. Then there exists $\Phi' \in \Psi_{\bullet,0}^{-\infty}(\mathcal{G})$ such that*

$$\text{id} - \Psi(\Phi + \Phi') \in \Psi_{\bullet,\infty}^{-\infty}(\mathcal{G}).$$

Proof. Let Φ be defined in Equation (14), φ be the reduced kernel of Φ . Then Theorem 4.5 implies that regarded as a kernel,

$$R := \text{id} - \psi \circ \varphi \in \Psi_{\varepsilon;\lambda}^{-\infty}(\mathcal{G}),$$

for some $\varepsilon > 0, \lambda > 0$. Using the arguments in the proof of Theorem 3.13 repeatedly, one has for any $k \in \mathbb{N}$,

$$\nabla^l \varphi \circ R^k(a) \leq M'_l M^{k\lambda} e^{-\varepsilon d(a, \mathbf{s}(a))} \hat{\rho}^{k\lambda},$$

for some constants M, M'_l . In particular, on an open neighborhood of \mathcal{G}_1 where $\hat{\rho}$ is sufficiently small,

$$\sum_{k=1}^N k \tilde{\nabla}^l(\tilde{\mathbf{m}}^*(\varphi \circ R^k))$$

converges uniformly and absolutely for all l . Define Φ' to be the limit

$$(15) \quad \Phi'(a) := \sum_{k=1}^{\infty} \theta(k\lambda \hat{\rho}(a))(\varphi \circ R^k)(a),$$

where $\theta \in C^\infty(\mathbb{R})$ is a function equal to 1 on a neighborhood of 0 with sufficiently small support. Observe that $\Phi' \in \Psi_{\varepsilon;0}^{-\infty}(\mathcal{G})$ since $\sum_k \theta(\hat{\rho}) M_l' M^{k\lambda} \hat{\rho}^{k\lambda}$ converges absolutely and uniformly. Moreover, for all $N = 1, 2, \dots$,

$$\Phi' - \sum_{k=1}^N \Phi R^k \in \Psi_{\varepsilon;N\lambda}^{-\infty}(\mathcal{G}).$$

Since $\text{id} - \Psi\Phi(\text{id} + R + R^2 + \dots + R^{N-1}) = R^N$, which lies in $\Psi_{\varepsilon;N\lambda}^{-\infty}(\mathcal{G})$ by Theorem 3.13, it follows that

$$\text{id} - \Psi(\Phi + \Phi') \in \bigcap_{N \in \mathbb{N}} \Psi_{\varepsilon;N\lambda}^{-\infty}(\mathcal{G}) = \Psi_{\varepsilon;\infty}^{-\infty}(\mathcal{G}).$$

□

Remark 4.7. By similar arguments, one gets $\tilde{\Phi} \in \Psi^{[-m]}(\mathcal{G}) \oplus \Psi_\varepsilon^{-\infty}(\mathcal{G})$ such that

$$\text{id} - \tilde{\Phi}\Psi \in \Psi_{\varepsilon;\infty}^{-\infty}(\mathcal{G}).$$

Remark 4.8. In the case of \mathcal{G} being uniformly non-degenerate, one may follow the arguments as in [10]. Note in that case the author needs an extra step to modify the parametrix Φ so that $(\text{id} - \Psi\Phi)^N$ is well defined for all N .

Theorem 4.9. *For any uniformly supported, elliptic operator $\Psi \in \Psi^{[m]}(\mathcal{G})$, suppose that Ψ is invertible. Then $\Psi^{-1} \in \Psi_\varepsilon^{-[m]}(\mathcal{G})$ for some $\varepsilon > 0$.*

Proof. Regard $\{\Psi_x\}_{x \in M_0}$ as a (pseudo-)differential operator on the manifold with bounded geometry M_0 , or in other words, a kernel on $M_0 \times M_0$. Then Lemma 4.4 again applies: Let ψ_0 be the kernel of $\Psi_x, x \in M_0$, ϕ be the reduced kernel of Ψ_x^{-1} . Then $\phi \in$

Let $\tilde{\Phi} := \Phi + \Phi'$ be defined in the previous lemma (Equation (15)) with reduced kernel $\bar{\varphi}$. Consider

$$\bar{\varphi}|_{\mathcal{G}_0} = \phi \circ (\psi|_{\mathcal{G}_0}) \circ (\bar{\varphi}|_{\mathcal{G}_0}) = \phi - \phi \circ (\kappa|_{\mathcal{G}_0}),$$

where κ is the reduced kernel of $\text{id} - \Psi(\Phi + \Phi')$.

We use similar arguments as in Lemma 3.14. Let $\Delta \in \Psi^{[m']}(\mathcal{G})$ be any elliptic differential operator with order $m' > m + \dim M_0$. Let $Q \in \Psi^{[-m']}(\mathcal{G})$ be a parametrix of Δ , $S := \text{id} - Q\Delta \in \Psi^{[m']}(\mathcal{G})$. Then one can write

$$\phi \circ (\kappa|_{\mathcal{G}_0}) = \phi(S|_{\mathcal{G}_0}) \circ \kappa + \phi(Q|_{\mathcal{G}_0}) \circ (\Delta\kappa)|_{\mathcal{G}_0}.$$

Since $\kappa \in \Psi_{\varepsilon;\infty}^{-\infty}(\mathcal{G})$, $\Delta\kappa \in \Psi_{\varepsilon;\infty}^{-\infty}(\mathcal{G})$. On the other hand, $\phi(Q|_{\mathcal{G}_0})$ is a uniform pseudo-differential operator of order less than $-\dim M_0$. Therefore by Lemma 4.4 $\phi(Q|_{\mathcal{G}_0})$ is continuous on $M_0 \times M_0$ and decays exponentially. It follows that the proof of 3.13 applies and $\phi(Q|_{\mathcal{G}_0}) \circ (\Delta\kappa)|_{\mathcal{G}_0} \in \Psi_{\varepsilon;\infty}^{-\infty}(\mathcal{G})$ (extending the kernel to \mathcal{G} by 0). The same argument holds for $\phi(S|_{\mathcal{G}_0}) \circ \kappa$. Hence we conclude that

$$(16) \quad \phi = \bar{\varphi} + \phi(S|_{\mathcal{G}_0}) \circ \kappa + \phi(Q|_{\mathcal{G}_0}) \circ (\Delta\kappa)|_{\mathcal{G}_0} \in \Psi^{[-m]}(\mathcal{G}) \oplus \Psi_\varepsilon^{-\infty}(\mathcal{G}).$$

□

4.3. The generalized inverse. Given a Fredholm operator T on a Hilbert space, it is a standard fact that both the null space and co-kernel of T are finite dimensional. Moreover T is invertible modulo projection onto its null space and co-kernel. In this section, let \mathcal{G} be uniformly degenerate, $\Psi \in \Psi^{[m]}(\mathcal{G})$, $m \geq 0$ elliptic, so that $\nu(\Psi) : \mathcal{W}^m(\mathcal{M}_0) \rightarrow \mathcal{L}^2(\mathcal{M}_0)$ is Fredholm. We describe the null space of $\nu(\Psi)$.

Let $G : \mathcal{L}^2(\mathcal{M}_0) \rightarrow \mathcal{W}^m(\mathcal{M}_0)$ be the generalized inverse of $\nu(\Psi)$. In other words,

$$(17) \quad \text{id} - G\nu(\Psi) = P_0, \quad \text{id} - \nu(\Psi)G = P_\perp,$$

where P_\perp and P_0 are the projection operators onto the null space and co-kernel of $\nu(\Psi)$ respectively.

To describe P_0 and P_\perp , we give an a-prior estimate of the null space of $\nu(\Psi)$ (and that of co-kernel of $\nu(\Psi)$) is similar.

Lemma 4.10. *Given any $f \in \mathcal{L}^2(\mathcal{M}_0)$ such that $\nu(\Psi)(f) = 0$. Then for any $f|_{\mathcal{M}_0}$ is smooth and for any $X_1, \dots, X_l \in \Gamma^\infty(\mathcal{A})$, $N \in \mathbb{N}$, $\rho^{-N} L_{\nu(X_1)} \circ \dots \circ L_{\nu(X_l)} f$ is bounded.*

Proof. Let $\tilde{\Phi}'$ be a parametrix of Ψ such that $R := \text{id} - \tilde{\Phi}'\Psi \in \Psi_{\bullet;\infty}^{-\infty}(\mathcal{G})$, as in Lemma 15. Then $\nu(R)f = f - \nu(\tilde{\Phi}')\nu(\Psi)f$. Since $\nu(R)f$ is smooth on \mathcal{M}_0 , f is smooth on \mathcal{M}_0 .

Moreover, for any $X_1, \dots, X_l \in \Gamma^\infty(\mathcal{A})$, $N \in \mathbb{N}$, let $S := L_{X_1} \circ \dots \circ L_{X_l} R$, then $S \in \Psi_{\varepsilon;\infty}^{-\infty}(\mathcal{G})$ for some $\varepsilon > 0$. Therefore for any N , $a \in \mathcal{G}$,

$$\begin{aligned} & |(\mathbf{t}^* \rho(a))^{-N} S(\mathbf{t}^* f)(a)| \\ & \leq (\mathbf{t}^* \rho(a))^{-N} M \int e^{-\varepsilon d(a,b)} ((\rho(\mathbf{t}(a)))^2 + (\rho(\mathbf{t}(b)))^2)^{\frac{N}{2}} (\mathbf{t}^* f)(b) \mu_{\mathbf{s}(a)}(b) \\ & \leq (\mathbf{t}^* \rho(a))^{-N} M' \int e^{-\varepsilon d(a,b)} (1 + e^{\frac{\varepsilon d(a,b)}{N}})^{\frac{N}{2}} (\mathbf{t}^* \rho(a))^N (\mathbf{t}^* f)(b) \mu_{\mathbf{s}(a)}(b), \end{aligned}$$

for some constants M, M' . Since by definition, we have $\mathcal{G}_0 = \mathcal{M}_0 \times \mathcal{M}_0$ and the \mathbf{s} -fiber is equipped with the same Riemannian density as \mathcal{M}_0 , it follows from the polynomial growth of \mathcal{G} and Cauchy-Schwartz inequality that the integral above is bounded independent of a . Hence the claim. \square

Define $\mathcal{S}(\mathcal{G}_0) \subset C^\infty(\mathcal{G}_0)$ to be the space of Schwartz functions on \mathcal{G}_0 with respect to ρ . In other words $\phi \in \mathcal{S}(\mathcal{G}_0)$ if and only if for all $l, N, N' \in \mathbb{N}$,

$$\begin{aligned} & \nabla^l \phi(d\tilde{\mathbf{m}}(X_1^{\mathbf{r}} \oplus Y_1^{\mathbf{r}}), \dots, d\tilde{\mathbf{m}}(X_l^{\mathbf{r}} \oplus Y_l^{\mathbf{r}}))(ab^{-1}) \\ & \leq M_{l;NN'} (\mathbf{t}^* \rho(ab^{-1}))^N (\mathbf{s}^* \rho(ab^{-1}))^{N'} \prod_{i=1}^l (|X|_i + |Y|_i), \end{aligned}$$

for some constants $M_{l;NN'} > 0$. Note that any functions on $\mathcal{S}(\mathcal{G}_0)$ extends to a smooth function on \mathcal{G} by 0. With such identification, we have for any $\varepsilon > 0$,

$$\Psi_{\varepsilon;\infty}^{-\infty}(\mathcal{G}) \subset \mathcal{S}(\mathcal{G}_0).$$

Theorem 4.11. *There exists $\Theta \in \Psi^{[-m]}(\mathcal{G}) \oplus \mathcal{S}(\mathcal{G}_0)$ such that $G = \nu(\Theta)$.*

Let κ_1, κ_2 be any two kernels in $\Psi_{\varepsilon;\infty}^{-\infty}(\mathcal{G})$. Observe that the vector representations $\nu(\kappa_1)$ on $\mathcal{L}^2(\mathcal{M}_0)$ is just convolution with $\kappa_1|_{\mathcal{G}_0}$. Using the identification $\mathcal{G}_0 \cong \mathcal{M}_0 \times \mathcal{M}_0$, one can write

$$\nu(\kappa_1)f(x) = \int \kappa_1|_{\mathcal{G}_0}(x, y)f(y)dy.$$

Observe that for any $y \in M_0$, $\kappa_1|_{s^{-1}(y)} = \kappa_1(\cdot, y) \in \mathcal{L}^2(M_0)$.

Let $T : \mathcal{L}^2(M_0) \rightarrow \mathcal{L}^2(M_0)$ be any bounded linear map. We claim that

Lemma 4.12. *The function $F : M_0 \times M_0 \rightarrow \mathbb{C}$*

$$F(x, y) := \int_{y_1 \in M_0} \kappa_1|_{t^{-1}(x)}(y_1)(T\kappa_2|_{s^{-1}(y)})(y_1)\mu_0(y_1)$$

is bounded and continuous.

Proof. With $l = 1$, and restrict to $\mathcal{G}_0 \cong M_0 \times M_0$, Definition 3.12 reduces to

$$d\kappa_2(x, y)(X \oplus Y) \leq M_1 e^{-\varepsilon_2 d(x, y)}(|X| + |Y|),$$

for any vector $X \oplus Y \in T_{(x, y)}(M_0 \times M_0)$. For any $y' \in M_0$ such that $t_0 := d(y, y')$ is smaller than the injectivity radius of M_0 , let $\gamma(t)$ be the unique geodesic joining y and y' . Then

$$|\kappa_2(y_2, y') - \kappa_2(y_2, y)| \leq M_1 e^{-\varepsilon_2(d(y_2, y) - t_0)} t_0.$$

Regard $\kappa_2(\cdot, y') - \kappa_2(\cdot, y)$ as a function in $\mathcal{L}^2(M_0)$ for each y, y' . By the boundedness of T , there is some $K > 0$ such that

$$\begin{aligned} \|T_{y_2}(\kappa_2(y_2, y') - \kappa_2(y_2, y))(\cdot)\|_{\mathcal{L}^2(M_0)}^2 &\leq K^2 \|(\kappa_2(\cdot, y') - \kappa_2(\cdot, y))\|_{\mathcal{L}^2(M_0)}^2 \\ &= K^2 \int |\kappa_2(y_1, y') - \kappa_2(y_1, y)|^2 \mu_0(y_1) \leq K^2 M_1^2 t_0^2 e^{2\varepsilon_2 t_0} \int e^{-2\varepsilon_2 d(y_1, y)} \mu_0(y_1). \end{aligned}$$

Since \mathcal{G} has polynomial growth by our assumption, the last integral $\int e^{-2\varepsilon_2 d(y_1, y)} \mu_0(y_1)$ is bounded by some constant C_2 , independent of y . Similarly,

$$\int |\kappa_1(x', y_1) - \kappa_1(x, y_1)|^2 \mu_0(y_1) \leq M_1'^2 t_0^2 e^{2\varepsilon_1 t_0} C_1.$$

for some constants $M_1', C_1 > 0$. It follows that

$$\begin{aligned} F(x', y') - F(x, y) &= \int (\kappa_1(x', y_1) - \kappa_1(x, y_1))(T_{y_2}(\kappa_2(y_2, y') - \kappa_1(y_2, y)))(y_1)\mu_0(y_1) \\ &\quad + \int \kappa_1(x, y_1)(T_{y_2}(\kappa_2(y_2, y') - \kappa_1(y_2, y)))(y_1)\mu_0(y_1) \\ &\quad + \int (\kappa_1(x', y_1) - \kappa_1(x, y_1))(T_{y_2}\kappa_2(y_2, y))(y_1)\mu_0(y_1) \\ &\leq (M_1' t_0 e^{\varepsilon_1 t_0} \sqrt{C_1})(M_1 t_0 e^{\varepsilon_2 t_0} \sqrt{C_2}) + \|\kappa_1(x, \cdot)\|_{\mathcal{L}^2(M_1 t_0 e^{\varepsilon_2 t_0} \sqrt{C_2})} \\ &\quad + (M_1' t_0 e^{\varepsilon_1 t_0} \sqrt{C_1}) \|\kappa_2(\cdot, y)\|_{\mathcal{L}^2}. \end{aligned}$$

It is clear that given any $(x, y) \in M_0 \times M_0$, the right hand side goes to 0 as $t_0 \rightarrow 0$. Hence $F(x, y)$ is continuous.

The proof of the boundedness of F is similar. We have $|\kappa_1(x, y_1)| \leq M_0' e^{-\varepsilon_1 d(x, y_1)}$, and $|\kappa_2(y_2, y)| \leq M_0 e^{-\varepsilon_1 d(y_2, y)}$, for some constants M_0, M_0' . Therefore

$$\begin{aligned} |F(x, y)| &\leq \|\kappa_1(x, \cdot)\|_{\mathcal{L}^2(M_0)} \|T_{y_2}\kappa_2(y_2, y)\|_{\mathcal{L}^2(M_0)} \\ &\leq \|\kappa_1(x, \cdot)\|_{\mathcal{L}^2(M_0)} K \|\kappa_2(\cdot, y)\|_{\mathcal{L}^2(M_0)} \leq K M_0' M_0 \sqrt{C_1 C_2}. \end{aligned}$$

□

We turn to the proof of Theorem 4.11.

Proof. (See [10, Theorem 4.20]) Let $\bar{\Phi}$ be defined in Equation (15) and $\tilde{\Phi}$ be as in Remark 4.7. Also, denote

$$\tilde{R} := \text{id} - \tilde{\Phi}\Psi, \bar{R} := \text{id} - \Psi\bar{\Phi}.$$

Computing $\tilde{\Phi}\Psi G$ and $G\Psi\bar{\Phi}$ in two different ways, one gets the equality

$$G = \tilde{\Phi}|_{\mathcal{G}_0} + \tilde{R}|_{\mathcal{G}_0} G - \tilde{\Phi}|_{\mathcal{G}_0} \circ P_{\perp} = \bar{\Phi}|_{\mathcal{G}_0} + G\bar{R}|_{\mathcal{G}_0} - P_0\tilde{\Phi}|_{\mathcal{G}_0}.$$

Rearranging, one gets

$$(18) \quad G = \tilde{\Phi}|_{\mathcal{G}_0} + \tilde{R}|_{\mathcal{G}_0} G\bar{R}|_{\mathcal{G}_0} + \tilde{R}|_{\mathcal{G}_0} \bar{\Phi}|_{\mathcal{G}_0} - \tilde{R}|_{\mathcal{G}_0} P_{\perp} \bar{\Phi}|_{\mathcal{G}_0} - \tilde{\Phi}|_{\mathcal{G}_0} P_0.$$

It is straightforward to see that

$$\tilde{R}|_{\mathcal{G}_0} P_{\perp} \bar{\Phi}|_{\mathcal{G}_0} \text{ and } \tilde{\Phi}|_{\mathcal{G}_0} P_0 \in \mathcal{S}(\mathcal{G}_0).$$

It remains to consider $\tilde{R}|_{\mathcal{G}_0} G\bar{R}|_{\mathcal{G}_0}$. From Lemma 4.12, it follows that $\tilde{R}|_{\mathcal{G}_0} G\bar{R}|_{\mathcal{G}_0}$ is given by convolution with some bounded continuous kernel ϕ on $M_0 \times M_0$. Using Equation (18) again, one gets

$$\begin{aligned} G &= \tilde{\Phi}|_{\mathcal{G}_0} + \tilde{R}|_{\mathcal{G}_0} (\tilde{\Phi}|_{\mathcal{G}_0} + \tilde{R}|_{\mathcal{G}_0} G\bar{R}|_{\mathcal{G}_0} + \tilde{R}|_{\mathcal{G}_0} \bar{\Phi}|_{\mathcal{G}_0} - \tilde{R}|_{\mathcal{G}_0} P_{\perp} \bar{\Phi}|_{\mathcal{G}_0} - \tilde{\Phi}|_{\mathcal{G}_0} P_0) \bar{R}|_{\mathcal{G}_0} \\ &\quad + \tilde{R}|_{\mathcal{G}_0} \bar{\Phi}|_{\mathcal{G}_0} - \tilde{R}|_{\mathcal{G}_0} P_{\perp} \bar{\Phi}|_{\mathcal{G}_0} - \tilde{\Phi}|_{\mathcal{G}_0} P_0. \end{aligned}$$

It is clear that $G - \tilde{\Phi}|_{\mathcal{G}_0} \in \mathcal{S}(\mathcal{G}_0)$. Hence $G - \tilde{\Phi}|_{\mathcal{G}_0} = \Theta_{-\infty}|_{\mathcal{G}_0}$ for some pseudo-differential operator $\Theta_{-\infty}$ of order $-\infty$ on \mathcal{G} . Finally we conclude that $G = \nu(\tilde{\Phi} + \Theta_{-\infty})$, and $\tilde{\Phi} + \Theta_{-\infty} \in \Psi^{[-m]}(\mathcal{G}) \oplus \mathcal{S}(\mathcal{G}_0)$. \square

4.4. The general case. To describe the inverse of a uniformly supported elliptic pseudo-differential operator on a general uniformly degenerate boundary groupoid $\mathcal{G} = \bigsqcup_{k=0}^r \mathcal{G}_k \times M_k \times M_k$, one repeats the arguments of Lemma 4.6 and Theorem 4.9. More precisely, it suffices to prove that

Theorem 4.13. *Let $\mathcal{G} = \bigsqcup_{k=0}^r \mathcal{G}_k \times M_k \times M_k$ be a uniformly degenerate boundary groupoid with smooth extension property. Given a uniformly supported elliptic pseudo-differential operator $\Psi = \{\Psi_x\}$ such that $\Psi|_{\mathcal{G}_k}$ are invertible for all $k \geq 1$. Suppose that for some $r' \leq r$ there $r - r'$ kernels $\{\varphi^{(k)}\}$, $k = r', \dots, r$, of the form $\varphi^{(k)} \sim \sum_{i=1}^r \varphi_i^{(k)}$, such that*

$$\text{id} - \psi \circ (\varphi + \varphi^{(r')} + \dots + \varphi^{(r)}) \in \Psi_{\bullet; \mathbf{0}_{r'}, \infty}^{-\infty}(\mathcal{G}),$$

where $\varphi_1^{(k)} \in C^\infty(\mathcal{G}) \cap \Psi_{\bullet; \mathbf{0}_{k+1}, \infty}^{-\infty}(\mathcal{G})$, and $\varphi_i^{(k)} \in \Psi_{\bullet; \mathbf{0}_k, \lambda_i^{(k)}, \infty}^{-\infty}(\mathcal{G})$, $\varphi_i^{(k)}|_{\mathcal{G} \setminus \bar{\mathcal{G}}_k} \in C^\infty(\mathcal{G} \setminus \bar{\mathcal{G}}_k)$, for all $i \geq 2$. Then

$$(i) \quad \Psi^{-1}|_{\bar{\mathcal{G}}_k} \in \Psi^{[-m]}(\bar{\mathcal{G}}_k) \oplus \Psi_{\bullet}^{-\infty}(\bar{\mathcal{G}}_k);$$

(ii) *There exists $\varphi^{(r')} \sim \sum_{i=1}^r \varphi_i^{(r')}$ where $\sum_{i=1}^r \varphi_i^{(r'-1)}$ satisfy the same smoothness and decaying conditions as above, such that*

$$\text{id} - \psi \circ (\varphi + \varphi^{(r'-1)} + \dots + \varphi^{(r)}) \in \Psi_{\bullet; \mathbf{0}_{r'-1}, \infty}^{-\infty}(\mathcal{G}).$$

Proof. To prove claim (i), let ϕ_k be the kernel of $(\Psi|_{\bar{\mathcal{G}}_k})^{-1}$ and consider $\phi_k \circ (\psi \circ (\varphi + \varphi^{(r')} + \dots + \varphi^{(r)}))|_{\bar{\mathcal{G}}_k}$ using the same arguments as Theorem 4.9. Moreover, by Equation (16),

$$\phi_k - (\varphi + \varphi^{(r')} + \dots + \varphi^{(r)})|_{\bar{\mathcal{G}}_k} \in \Psi_{\bullet; \infty}^{-\infty}(\bar{\mathcal{G}}_k).$$

Using the smooth extension property, let $\hat{\varphi}_{r'-1} \in \Psi_{\bullet; \mathbf{0}_{r'-1}, \infty}^{-\infty} \cap C^\infty(\mathcal{G})$ be such that

$$\text{id} - \psi \circ (\varphi + \hat{\varphi}_{r'-1} + \varphi^{(r')} + \cdots + \varphi^{(r)}) \in \Psi_{\bullet; \mathbf{0}_{r'-1}, \lambda, \infty}^{-\infty}(\mathcal{G}).$$

Then the arguments of Lemma 4.6 can be applied to prove (ii). Note that the Neumann series is finite on compact subsets and hence the limit is in $C^\infty(\mathcal{G} \setminus \bar{\mathcal{G}}_{k-1})$. \square

5. CONCLUDING REMARKS

In this paper, we constructed a rather complete analogue of the big and full calculus to [10], namely, the exponentially decaying calculus and a finer space of kernels with asymptotic expansions. We proved that these spaces are filtered like the full calculus, and contains the compact parametrices and generalized inverse of elliptic differential operators.

We remark that the definition of boundary groupoids and uniformly degenerate operators we considered is somewhat restricted. For instance, it would seem to be rather obvious to generalize to the notion of boundary groupoids to contain invariant sub-manifolds of the form $G_k \times M \times_B M$. Also, proving conjecture 4.3 would be a major advancement of the theory.

The full calculus constructed in this paper should enable one to re-write many classical results in the groupoid context. On the more geometrical side, some construction had been exemplified in [17]. There, the author considers the heat kernel of generalized Laplacian operators and constructs renormalized index for the Bruhat sphere. One should be able to generalize the results in [17] with the framework constructed here. In particular, the functions ρ_k can be used as regularizing functions. In the same vein, complex powers of elliptic operators, as well as holomorphic functional calculus of groupoid pseudo-differential operators, are also very interesting directions for future research.

On the side of more traditional analysis, one would study boundary problems involving (vector representations of) groupoid differential operators, or even non-linear equations.

APPENDIX A. MANIFOLDS WITH BOUNDED GEOMETRY

In this section, we recall the definition of manifolds with bounded geometry and some classes of functions and operators defined on it. For details, see [16].

Definition A.1. A Riemannian manifold M is said to have bounded geometry if

- (i) M has positive injectivity radius;
- (ii) The Riemannian curvature R of M has bounded covariant derivatives.

Lemma A.2. [16, Lemma 1.2] *There exists $r_0 > 0$ such that for any $0 < r < r_0$, there is a countable set $\{x_\alpha\} \subset M$ such that the balls $B(x_\alpha, \varepsilon)$ is a cover of M , and any $x \in M$ belongs to at most N balls $B(x_\alpha, 2r)$, for some N independent of x .*

Lemma A.3. *Let $\{(B(x_\alpha, \varepsilon), \mathbf{x}_\alpha)\}$ be a cover by normal coordinates patches, such that the conclusion of Lemma A.2 holds. Then there exists a partition of unity θ_α subordinated to $\{B(x_\alpha, \varepsilon)\}$, such that for any $k \in \mathbb{N}$, all k -th order partial derivatives of θ_α are bounded by some C_k , independent of α .*

For each $m \in \mathbb{R}$, define the 2-norms

$$(19) \quad \|f\|_{2,m} := \left(\sum_{\alpha} \|\theta_{\alpha} f\|_{\mathcal{W}^m(U_{\alpha})}^2 \right)^{\frac{1}{2}},$$

where $\mathcal{W}^m(U_{\alpha})$ is the m -th Sobolev norm on $U_{\alpha} \subset \mathbb{R}^n$. We denote the completion of $C_c^{\infty}(M_0)$ with respect to $\|\cdot\|_{2,m}$ by $\mathcal{W}^m(M)$.

Observe that, since all transition functions are uniformly bounded, the equivalence classes of these norms are independent of the choices made.

On a manifold with bounded geometry, a class of ‘uniformly bounded’ pseudo-differential operators can also be defined. Fix any covering $\{U_{\alpha}, \mathbf{x}_{\alpha}\}$ of M by normal coordinates. Let $\Psi \in \psi_{\varrho}^m(M)$. Recall that $(\mathbf{x}_{\alpha}^{-1})^* \psi \mathbf{x}_{\alpha}^*$ is a pseudo-differential operator on U_{α} . Let $\sigma_{\alpha} \in \mathbf{S}^m(U_{\alpha})$ be the total symbol of $(\mathbf{x}_{\alpha}^{-1})^* \psi \mathbf{x}_{\alpha}^*$. Then we say that

Definition A.4. The pseudo-differential operator Ψ is *uniformly bounded* if

- (i) The support of Ψ is contained in the set

$$\{(x, y) \in M \times M : d(x, y) < r\}$$

for some $r > 0$;

- (ii) For any multi-indexes I, J , there exists a constant C_{IJ} , independent of α , such that

$$|\partial_x^I \partial_{\zeta}^J \sigma_{\alpha}| \leq C_{IJ} (1 + |\zeta|)^{m-|J|}.$$

We denote the set of all, uniformly bounded pseudo-differential operators of order $\leq m$ by $\Psi_b^m(M)$.

APPENDIX B. PROOF OF THEOREM 4.2

We consider the special case when $\mathcal{G} = M_0 \times M_0 \sqcup G \times M_1 \times M_1$. Let $p = \dim G, q = \dim M_1$. For simplicity, we denote the only defining function by ρ .

B.1. The exponential map. First, recall the definition of admissible section and exponential map of a groupoid.

Definition B.1. An admissible section is a smooth map $S : M \rightarrow \mathcal{G}$ such that $\mathbf{s} \circ S = \text{id}$ and $\mathbf{t} \circ S$ is a diffeomorphism on M .

One has a semi-group structure on the set of all admissible sections defined by

$$S_1 S_2(x) := S_1(\mathbf{t} \circ S_2(x)) S_2(x),$$

where the right hand side is the groupoid multiplication. Likewise, each admissible section S induces a diffeomorphism on \mathcal{G} given by

$$aS := aS((\mathbf{t} \circ S)^{-1}(a)).$$

It is easy to see that $(aS_1)S_2 = a(S_1 S_2)$ for any admissible sections S_1, S_2 .

Remark B.2. In the special case when $\mathcal{G} = G$ is a Lie group, $Z \mapsto \exp Z(e)$ is just the Lie group exponential map.

Given any smooth section $X \in \Gamma^{\infty}(A)$, denote by $X^{\mathbf{r}}$ the right invariant vector field on \mathcal{G} with $\mathbf{s}^* X^{\mathbf{r}} = 0$ and $X^{\mathbf{r}}|_M = X$. Since M is compact, it is standard that $X^{\mathbf{r}}$ is a complete vector field on \mathcal{G} , hence one has a well defined map

$$\exp X : M \rightarrow \mathcal{G},$$

given by the flow of $X^\mathbf{r}$ from each $x \in M \subset \mathcal{G}$. It is a well known fact that $\mathbf{t} \circ \exp X$ equals the flow of $\nu(X)$ on M and hence is a $\exp X$ is an admissible section. Define

$$E_X := d\mathbf{t} \circ d(\exp X|_A) : A \rightarrow A.$$

We list some basic properties of the exponential map [13], [8]:

- (i) For any $X, Y \in C^\infty(A)$, $\exp X \exp Y = \exp Y \exp E_X$;
- (ii) For any $x \in M$, $((\exp X)(x))^{-1} = \exp(-X)(E_X^\nu(x))$, where $E_X^\nu : M \rightarrow M$ is the flow of $\nu(X)$.

Notation B.3. For any collection of sections $Z_I = (Z_1, \dots, Z_{|I|}) \in \Gamma^\infty(A)$, denote

$$\exp Z_I := \exp Z_{|I|} \exp Z_{|I|-1} \cdots \exp Z_2 \exp Z_1.$$

For any $\mu = (\mu_1, \dots, \mu_{|I|}) \in \mathbb{R}^{|I|}$, denote

$$\exp(\mu \cdot Z_I) := \exp \mu_{|I|} Z_{|I|} \exp \mu_{|I|-1} Z_{|I|-1} \cdots \exp \mu_2 Z_2 \exp \mu_1 Z_1.$$

We adapt the construction of exponential coordinates charts on a groupoid in [13] to our case.

Lemma B.4. *Let $Z_I \subset \Gamma^\infty(A)$, U_α coordinates patch of M . Let $X_1^{(\alpha)}, \dots, X_k^{(\alpha)}$ be a local basis over $(\mathbf{t} \circ \exp Z_I)(U_\alpha)$. Then there exists $\delta > 0$ such that the map $\mathbf{x}_{Z_I}^{(\alpha)} : (-\delta, \delta)^k \times U_\alpha \rightarrow \mathcal{G}$,*

$$\mathbf{x}_{Z_I}^{(\alpha)}(\tau, x) := \exp(\tau \cdot (X_1^{(\alpha)}, \dots, X_k^{(\alpha)})) \exp Z_I(x),$$

is a diffeomorphism onto its image.

B.2. Exponential coordinates on $(M_0 \times M_0) \sqcup (G \times M_1 \times M_1)$. We turn to our special case when $\mathcal{G} = (M_0 \times M_0) \sqcup (G \times M_1 \times M_1)$. First consider exponential coordinates on G .

Lemma B.5. *Let (Y_1, \dots, Y_p) be a fixed basis of \mathfrak{g} . There exists a cover of G by coordinates patches of the form*

$$(-\delta, \delta)^p \ni \mu \mapsto \exp(\mu \cdot (Y_1, \dots, Y_p)) \exp(Z_I^G)(e),$$

for some collections $Z_I^G \in \mathfrak{g}$, such that

- (i) *The cover is locally finite with uniformly bounded index;*
- (ii) *$|Z_1^G|, \dots, |Z_{|I|}^G| \leq r_{\mathfrak{g}}$, for some $r_{\mathfrak{g}} > 0$ (independent of I);*
- (iii) *There exist a constant $C_G > 0$ such that*

$$d(e, \exp Z_I^G) > C_G(|I| - 1),$$

where $d(\cdot, \cdot)$ is the right invariant metric on G ;

Proof. For any $r > 0$, denote by $B_G(g, r)$ and $B_{\mathfrak{g}}(0, r)$ the ball on G (resp. \mathfrak{g}) of radius r centered at $g \in G$ (resp. $0 \in \mathfrak{g}$).

Let $r > 0$ be such that $B(e, 2r) \subseteq \{\exp(\mu \cdot (Y_1, \dots, Y_p)(e)) : \mu \in (-\delta, \delta)^p\}$. Take a maximal collection of subset of the form

$$B_G(g_i, r) = \{gg_i : g \in B_G(e, r)\}.$$

Since G is a manifold with bounded geometry, it is standard that $\{B_G(g_i, 2r)\}$ is a covering satisfying condition (i), and hence the covering

$$\{\exp(\mu \cdot (Y_1, \dots, Y_p)(e))g_i : \mu \in (-\delta, \delta)^p\}.$$

It remains to find for each i , $g_i = \exp Z_{I_i}^G$, some $Z_{I_i}^G$ satisfying condition (ii). It is elementary that there exists a constant $r_{\mathfrak{g}} > 0$ such that the exponential map

$$\exp : B_{\mathfrak{g}}(0, r_{\mathfrak{g}}) \rightarrow G$$

is a diffeomorphism onto its image. Moreover, one has $\exp(B_{\mathfrak{g}}(0, r_{\mathfrak{g}})) \supseteq B_G(e, C_G)$ for some constants $C_G > 0$.

Let $\gamma(t)$ be a unit speed minimizing geodesic joining e and g_i . Parameterize γ so that $\gamma(0) = e, \gamma(L) = g_i$. Then $L = d(g_i, e)$. Define $g_l := \gamma(C_G l), l = 0, 1, \dots, L'$, where L' is the largest integer such that $C_G L' \leq L$. Then $g = g g_{L'}^{-1} g_{L'} \cdots g_1 g_0^{-1} g_0$. By right invariance $g_l g_{l-1}^{-1} \in B_G(e, r_{\mathfrak{g}})$ for any l . Therefore by definition there exists $Z_l \in \mathfrak{g}, 1 \leq l \leq L' + 1$, such that $|Z_l^G| < r_{\mathfrak{g}}$ and

$$\exp Z_{L'+1}^G = g g_{L'}^{-1}, \quad \exp Z_l^G = g_l g_{l-1}^{-1} \quad \forall l \leq L'.$$

Let $Z_{I_i}^G := \{Z_1^G, \dots, Z_{L'+1}^G\}$. It is clear that the collection $Z_{I_i}^G$ satisfies conditions (ii) and (iii). \square

Let $\{Y_1^{\mathfrak{g}}, \dots, Y_p^{\mathfrak{g}}\}$ be an orthonormal basis of \mathfrak{g} . Regard $\{Y_1^{\mathfrak{g}}, \dots, Y_p^{\mathfrak{g}}\}$ as a basis of $M_1 \times \mathfrak{g}$. Let $\{Y_1, \dots, Y_p\}$ be an extension of $\{Y_1^{\mathfrak{g}}, \dots, Y_p^{\mathfrak{g}}\}$ to $\Gamma^\infty(A)$, such that $\{Y_1, \dots, Y_p\}$ is a local orthonormal basis on some open set $U \supset M_1$ of M .

Lemma B.6. *Given any collection $Z_I = (Z_1, \dots, Z_{|I|}) \subset \text{Span}_{\mathbb{R}}\{Y_1, \dots, Y_p\}$, $|Z_m| < r_{\mathfrak{g}}$ for all $1 \leq m \leq |I|$. For any $M > 0$, there exists $r > 0$ such that*

$$d(x, \mathbf{t} \circ \exp Z_I(x)) \leq M,$$

whenever $x \in B(M_1, e^{-\omega r_{\mathfrak{g}}|I|r})$.

Proof. Let ρ be a smooth function on $M \setminus M_1$, such that $\rho = d(\cdot, M_1)$, as in Lemma 3.1. Since for any $Z \in \text{Span}_{\mathbb{R}}\{Y_1, \dots, Y_p\}$, $\nu(Z) = 0$ on M_1 and $|\nu(Z)|$ is a Lipschitz function, there exists $M' > 0$ such that

$$(20) \quad |\nu(Z)(x)| \leq M' \rho(x),$$

for any $Z \in \text{Span}_{\mathbb{R}}\{Y_1, \dots, Y_p\}, |Z| \leq r_{\mathfrak{g}}$.

Write $x_0 := x, x_1 := \exp Z_{i-1} \cdots \exp Z_1(x), i = 1, \dots, |I|$. Then x_{i-1}, x_i is joined by the curve $(\mathbf{t} \circ \exp t Z_i)(x_{i-1}), t \in [0, 1]$, whose length is

$$\begin{aligned} \int_0^1 |\nu(Z_i)((\mathbf{t} \circ \exp t Z_i)(x_{i-1}))| dt &\leq \int_0^1 M' \rho((\mathbf{t} \circ \exp t Z_i)(x_{i-1})) dt \\ &\leq \int_0^1 M' e^{\omega r_{\mathfrak{g}}(i+1)} \rho(x) dt, \end{aligned}$$

where we used Theorem 3.3 for the last inequality. Hence by the triangular inequality

$$d(x, \mathbf{t} \circ \exp Z_I(x)) \leq \frac{M' e^{\omega r_{\mathfrak{g}}} (e^{\omega r_{\mathfrak{g}}|I|} - 1) \rho(x)}{e^{\omega r_{\mathfrak{g}}} - 1}.$$

The claim then follows by putting $r_0 \leq \frac{M(e^{\omega r_{\mathfrak{g}}} - 1)}{M' e^{\omega r_{\mathfrak{g}}}}$ and such that $\rho = d(\cdot, M_1)$ on $B(M_1, r_0)$. \square

Let $L > 0$ be such that the injectivity radius of M_1 is greater than $2L$. Then M_1 can be covered by a finite collection of balls $\{B_{M_1}(x_\alpha, L)\}$. Let $\mathbf{x}_{M_1}^{(\alpha)}$ be a local

coordinates chart of $B_{M_1}(x_\alpha, 2L)$. Fix a trivialization of $TM_1^\perp|_{B_{M_1}(x_\alpha, 2L)}$ for each α and let \tilde{U}^α be the coordinate patches

$$\exp_{\mathbf{x}_{M_1}^{(\alpha)}(x_1, \dots, x_q)}(x_{p+1}, \dots, x_n),$$

where \exp here denotes the Riemannian exponential map.

Fix an orthonormal basis $\{Y_1, \dots, Y_p\}$ of \mathfrak{g} . Regard it as a basis of $TM_1 \times \mathfrak{g}$ and extend to an orthonormal set of sections on $A|_{\bigcup \tilde{U}_\alpha}$. We still denote the extension by $\{Y_1, \dots, Y_p\}$. It is then a standard construction that there exists

- A finite set of collections of sections $X_J \subset \Gamma^\infty(A)$;
- for each α, J , a basis $X_1^{\alpha, J}, \dots, X_p^{\alpha, J} \in \Gamma^\infty(\exp X_J(\tilde{U}_\alpha))$,

such that

- (i) the local coordinates $\tau \mapsto \exp(\tau \cdot (X_1^{\alpha, J}, \dots, X_p^{\alpha, J})) \exp X_J(x)$, $\tau \in (-\delta, \delta)^p$, $x \in U_\alpha \cap M_1$, is an atlas of $M_1 \times M_1$;
- (ii) On $\exp X_J(\tilde{U}_\alpha)$, $\{Y_1, \dots, Y_p, X_1^{\alpha, J}, \dots, X_q^{\alpha, J}\}$ is an orthonormal basis of $A|_{\exp X_J(\tilde{U}_\alpha)}$.

It follows that for some $r > 0$, the map $\mathbf{x}_{Z_I, X_J}^\alpha : (-\delta, \delta)^{p+q} \times (U_\alpha \cap B(M_1, e^{-\omega r_\mathfrak{g}|I|_r})) \rightarrow \mathcal{G}$ defined by

$$(21) \quad \mathbf{x}_{X_J, Z_I}^{(\alpha)}(\mu, \tau, x) \\ := \exp(\mu \cdot (Y_1, \dots, Y_p)) \exp(\tau \cdot (X_1^{\alpha, I}, \dots, X_p^{\alpha, I})) \exp X_J \exp Z_I(x)$$

satisfies the conditions of Lemma B.4 and hence defines a local coordinates patch in \mathcal{G} . Moreover, $\{U_{X_J, Z_I}^\alpha \cap (G \times M_1 \times M_1)\}$ is an atlas of $G \times M_1 \times M_1$.

Recall that $\tilde{\mathcal{G}} := \{(a, b) \in \mathcal{G} \times \mathcal{G} : \mathbf{s}(a) = \mathbf{s}(b)\}$ and one has the map $\tilde{\mathbf{m}} : \tilde{\mathcal{G}} \rightarrow \mathcal{G}$ defined by $\tilde{\mathbf{m}}(a, b) := ab^{-1}$. Consider writing $d\tilde{\mathbf{m}}(V^\mathbf{r} \oplus W^\mathbf{r})$ on the coordinates chart $(\mathbf{x}_{X_J, Z_I}^{(\alpha)}, U_{X_J, Z_I}^{(\alpha)})$.

By Equation (5), it is straightforward to compute for any $a \in U_{X_J, Z_I}^{(\alpha)}$,

$$\partial_1(a) = Y_1^\mathbf{r}(a), \partial_2 = (E_{Y_1} Y_2)^\mathbf{r}(a), \dots$$

and so on. It follows that on the coordinates chart

$$\mathbf{x}_\emptyset^{(\alpha)}(\mu', \tau', x') := \exp(\tau \cdot (X_1^{\alpha, I}, \dots, X_p^{\alpha, I}))(E_{Z_I}^{-1} \circ E_{X_J}^{-1}(x)),$$

if one writes $V^\mathbf{r} = \sum_{i=1}^n v_i(x') \partial_{(\mu', \tau')_i}$ on $U_\emptyset^{(\alpha)}$ (Note that there is no $\partial_{x'_i}$ since $V^\mathbf{r}$ is tangential to the \mathbf{s} -fibers), then $V^\mathbf{r} = \sum_{i=1}^n v_i(E_{X_J} E_{Z_I}(x)) \times \partial_{(\mu, \tau)_i}$ on $U_{X_J, Z_I}^{(\alpha)}$.

We turn to consider the case $d\tilde{\mathbf{m}}(0 \oplus W^\mathbf{r})$. For any $a = \mathbf{x}_{X_J, Z_I}^{(\alpha)}(\tau, \mu, x)$, one has

$$(22) \quad d\tilde{\mathbf{m}}(0 \oplus W^\mathbf{r})(a) \\ = \partial_t \Big|_{t=0} \exp(-t E_{\mu_1 Y_1} \circ \dots \circ E_{\mu_p Y_p} \circ E_{\tau_1 X_1^{(\alpha)}} \circ \dots \circ E_{\tau_q X_q^{(\alpha)}} \circ E_{X_J} \circ E_{Z_I} W) \\ \exp(\tau \cdot (X_1^{(\alpha)}, \dots, X_q^{(\alpha)})) \exp(\mu \cdot (Y_1, \dots, Y_p)) \exp X_J \exp Z_I(E_{tW}^\nu(x)) \\ = (E_{\mu_1 Y_1} \circ \dots \circ E_{\mu_p Y_p} \circ E_{\tau_1 X_1^{(\alpha)}} \circ \dots \circ E_{\tau_q X_q^{(\alpha)}} \circ E_{X_J} \circ E_{Z_I} W)^\mathbf{r}(a) \\ + d(\exp(\tau \cdot (X_1^{(\alpha)}, \dots, X_q^{(\alpha)})) \exp(\mu \cdot (Y_1, \dots, Y_p)) \exp_{X_J} \exp_{Z_I}) \nu(W)(\mathbf{s}(a)).$$

To proceed, we estimate $E_{Z_I} W$.

Lemma B.7. *There exists constants $K, r > 0$, such that for any $Z_I = \{Z_1, \dots, Z_I\} \subset \text{Span}_{\mathbb{R}}\{Y_1, \dots, Y_p\}, |Z_1|, \dots, |Z_I| \leq r_{\mathfrak{g}}$,*

(i) $B(M_1, r) \subset \bigcup_{\alpha} U_{\alpha}$;

(ii) *For any $W \in A|_{B(M_1, e^{-\omega r_{\mathfrak{g}}|I|}r)}$ $|E_{Z_I} \circ \dots \circ E_{Z_1} W| \leq K e^{C(\log |I|)^2}$,*

where $\omega > 0$ is such that Equation (2) is satisfied on $B(M_1, r)$.

Proof. Only estimate (ii) is not obvious. For each α , define $P^{\alpha} : U_{\alpha} \rightarrow M_1 \cap U_{\alpha}$ to be the coordinates projection. For each $x \in \tilde{U}_{\alpha}$, define $T_a^{\alpha} : A|_{\tilde{U}_{\alpha}} \rightarrow A_x$ to be the natural projection by identifying $A|_{\tilde{U}_{\alpha}} \cong \mathbb{R}^{p+q} \times \tilde{U}_{\alpha}$, using the basis $\{Y_1, \dots, Y_p, X_1^{(\alpha)}, \dots, X_q^{(\alpha)}\}$.

Define the functions $E^{\alpha} : \text{Span}_{\mathbb{R}}\{Y_1, \dots, Y_p\} \times A|_{U_{\alpha}} \rightarrow A|_{\tilde{U}_{\alpha}}$,

$$E_Z^{\alpha} W := T_{\Phi_Z^{\alpha}(x)}^{\alpha} \circ E_Z \circ T_{P^{\alpha}(x)}^{\alpha}(W), \quad W \in A_x,$$

and $F^{\alpha} := E - E^{\alpha}$.

We analyze E^{α} . Let $W_0 := W \in A_{x_0}$ and $W_i := E_{Z_i}^{\alpha} \circ \dots \circ E_{Z_1}^{\alpha} W \in A_{x_i}$. Define $P_{\mathfrak{g}} : A|_{M_1} \rightarrow \mathfrak{g}$ to be the natural projection, then one has

$$(23) \quad P_{\mathfrak{g}}(T_{P^{\alpha}(x_{i+1})}^{\alpha} E^{\alpha}(W_i)) = \text{Ad}_{\exp Z_i}(P_{\mathfrak{g}}(T_{P^{\alpha}(x_i)}^{\alpha}(W_i))).$$

Iterating Equation (23), one gets

$$(24) \quad P_{\mathfrak{g}}(T_{P^{\alpha}(x_m)}^{\alpha}(W_m)) = \text{Ad}_{\exp Z_m} \circ \dots \circ \text{Ad}_{\exp Z_1} P_{\mathfrak{g}}(T_{P^{\alpha}(x_0)}^{\alpha}(W)).$$

Since $\{X^{(\alpha)}, Y\}$ are orthonormal bases, T_x^{α} is an isometry for any x . Hence, using Equation (24) and the fact that E_{Z_i} acts as identity on TM_1 , one gets

$$\begin{aligned} |W_m| &= |E_{Z_i}^{\alpha} \circ \dots \circ E_{Z_1}^{\alpha} W| \\ &= \left(\left| \text{Ad}_{\exp Z_m} \circ \dots \circ \text{Ad}_{\exp Z_1} P_{\mathfrak{g}}(T_{P^{\alpha}(x_0)}^{\alpha}(W)) \right|^2 + \left| (\text{id} - P_{\mathfrak{g}})(T_{P^{\alpha}(x_0)}^{\alpha}(W)) \right|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Using the assumption that G is nilpotent, one can find a constant N_G such that

$$|\text{Ad}_{\exp Z'_I} \circ \dots \circ \text{Ad}_{\exp Z'_1}| \leq K_0 |I|^{N_G},$$

for any collection $Z'_I = \{Z'_1, \dots, Z'_I\} \subset B$. Then it is clear that

$$(25) \quad |E_{Z_m}^{\alpha} \circ \dots \circ E_{Z_1}^{\alpha} W| \leq K_1 m^{N_G} |W|.$$

We turn to F^{α} . Observe that $F_Z^{\alpha} W = 0$ for any $\alpha, W \in A|_{M_1}$. Regard F_Z^{α} as a matrix valued function on $U_{\alpha} \cap B(M_1, r)$. Then the differentiability of F_Z^{α} implies there exists $K_2 > 0$ such that

$$|F_Z^{\alpha} W| \leq K_2 d(x, M_1) |W|, \quad \forall W \in A_x, x \in U_{\alpha} \cap B(M_1, r).$$

Now we return to $E_{Z_m} \circ \dots \circ E_{Z_1} W$. We expand

$$\begin{aligned} (26) \quad E_{Z_m} \circ \dots \circ E_{Z_1} W &= E_{Z_m}^{\alpha} \circ E_{Z_{m-1}}^{\alpha} \circ \dots \circ E_{Z_1}^{\alpha} W + F_{Z_m}^{\alpha} \circ E_{Z_{m-1}}^{\alpha} \circ \dots \circ E_{Z_1}^{\alpha} W \\ &\quad + E_{Z_m}^{\alpha} \circ F_{Z_{m-1}}^{\alpha} \circ \dots \circ E_{Z_1}^{\alpha} W + F_{Z_m}^{\alpha} \circ F_{Z_{m-1}}^{\alpha} \circ \dots \circ E_{Z_1}^{\alpha} W \\ &\quad + \dots + F_{Z_m}^{\alpha} \circ F_{Z_{m-1}}^{\alpha} \circ \dots \circ F_{Z_1}^{\alpha} W. \end{aligned}$$

From Equation (25) and our construction of F^α , it is straightforward to estimate that each term of the right hand side of Equation (26) is bounded by

$$(K_2 r)^{m'} e^{-\frac{\omega r_{\mathfrak{g}}(m'^2+m')}{2}} (K_1 m^{N_G})^{m'+1},$$

where m' is the number of F^α . One then adds all terms in the right hand side of Equation (26) together and gets the estimate

$$\begin{aligned} |E_{Z_m} \circ \cdots \circ E_{Z_1} W| &\leq |W| \sum_{m'=0}^m \frac{m!}{m'!(m-m')!} (K_2 r)^{m'} e^{-\frac{\omega r_{\mathfrak{g}}(m'^2+m')}{2}} (K_1 m^{N_G})^{m'+1} \\ &\leq |W| \sum_{m'=0}^m m^{m'} (K_2 r)^{m'} e^{-\frac{\omega r_{\mathfrak{g}}(m'^2+m')}{2}} (K_1 m^{N_G})^{m'+1}. \end{aligned}$$

We split the above sum into two: the first from $m' = 0$ to $m' = N-1$, and the second from $m' = N$ to $m' = m$, where N is the smallest positive integer such that

$$(K_2 r) e^{-\frac{\omega r_{\mathfrak{g}} N}{2}} (K_1 m^{N_G+1}) < \frac{1}{2},$$

in other words, $N > \frac{2 \log(2K_2 r K_1 m^{N_G+1})}{\omega r_{\mathfrak{g}}}$. Then one has

$$|W| \sum_{m'=N}^m m^{m'} K_1 m^{N_G} (K_2 r e^{-\frac{\omega r_{\mathfrak{g}}(m'+1)}{2}} K_1 m^{N_G})^{m'} \leq \frac{|W| K_1 m^{N_G}}{2^{N-1}},$$

by assumption. On the other hand,

$$\begin{aligned} |W| \sum_{m'=0}^{N-1} K_1 m^{N_G} (K_2 r e^{-\frac{\omega r_{\mathfrak{g}}(m'+1)}{2}} K_1 m^{N_G+1})^{m'} \\ \leq |W| \sum_{m'=0}^{N-1} K_1 m^{N_G} (K_2 r e^{-\frac{\omega r_{\mathfrak{g}}}{2}} K_1 m^{N_G+1})^{m'} \\ = \frac{|W| K_1 m^{N_G} ((K_2 r e^{-\frac{\omega r_{\mathfrak{g}}}{2}} K_1 m^{N_G+1})^N - 1)}{(K_2 r e^{-\frac{\omega r_{\mathfrak{g}}}{2}} K_1 m^{N_G+1}) - 1}. \end{aligned}$$

Observe that

$$m^N \leq K_3 m^{N' \log m} = K_3 e^{N'(\log m)^2},$$

for some constants K_3, N' . Therefore the estimation (ii) follows. \square

B.3. An exponentially decaying extension. In this section, fix a coordinates cover as defined in Equation (21). Let $\theta_{Z_I}^G$ be a partition of unity of G subordinated to $B(\exp Z_I, r)$, and $\theta_{X_J}^\alpha$ be a partition of unity of M_1 subordinated to $U_\alpha \cap M_1$. Let $\theta \in C_c^\infty(\mathbb{R})$ be such that χ equals 1 on $(-\infty, 1)$ and 0 on $(2, \infty)$.

Given any $\psi \in \Psi_\varepsilon^\infty(\mathcal{G}_1)$, define $\theta_{X_J, Z_I}^\alpha \in C_c^\infty(U_{X_J, Z_I}^{(\alpha)})$ by

$$\begin{aligned} \theta_{X_J, Z_I}^\alpha(\mathbf{x}_{X_J, Z_I}^{(\alpha)}(\tau, \mu, x)) &:= \theta_{X_J}^\alpha(\exp(\tau \cdot (X_1^{\alpha, J}, \dots, X_p^{\alpha, J})) \exp X_I)(P^\alpha(x)) \\ &\quad \times \theta_{Z_I}^G(\exp(\mu \cdot (Y_1, \dots, Y_p) \exp Z_I(e))(P^\alpha(x)) \\ &\quad \times \theta(2e^{\omega r_{\mathfrak{g}}|I|} r^{-1} \rho(x)). \end{aligned}$$

Here, recall that $P^\alpha : U^\alpha \rightarrow U^\alpha \cap M_1$ is the coordinates projection. Given any $\psi \in \Psi_{\varepsilon; \mathbf{0}}^\infty(\mathcal{G}_1)$, $\varepsilon > 0$, let

$$(27) \quad \bar{\psi} := \sum_{X_J, Z_I} \psi_{X_J, Z_I}^{(\alpha)},$$

where $\psi_{X_J, Z_I}^{(\alpha)} \in C_c^\infty(U_{X_J, Z_I}^{(\alpha)})$ is defined by

$$\psi_{X_J, Z_I}^{(\alpha)}(\mathbf{x}_{X_J, Z_I}^{(\alpha)}(\tau, \mu, x)) := \theta_{X_J, Z_I}^\alpha(\mathbf{x}_{X_J, Z_I}^{(\alpha)}(\tau, \mu, x))\psi(\mathbf{x}_{X_J, Z_I}^{(\alpha)}(\tau, \mu, P^\alpha(x)),$$

i.e., by extending some cutoff of ψ along coordinate curves. We claim that

Proposition B.8. *The sum in Equation (27) converges absolutely and the kernel $\bar{\psi} \in \Psi_{\varepsilon r_{\mathfrak{g}}^{-1} C_G; \mathbf{0}}^{-\infty}(\mathcal{G})$.*

Proof. Given each U_{X_J, Z_I}^α , consider $d(a, \mathbf{s}(a))$ and $d(b, \mathbf{s}(b))$ for any $a \in U_{X_J, Z_I}^\alpha, b \in U_{X_J, Z_I}^\alpha \cap (\mathcal{G}_1)$. By construction, there is a path of length $\leq |I|r_{\mathfrak{g}} + C'$ joining a and $\mathbf{s}(a)$, for some constant C' independent of X_J, Z_I . It follows that $d(a, \mathbf{s}(a)) \leq |I|r_{\mathfrak{g}} + C'$. On the other hand, by (3) of Lemma B.5 one has $d(b, \mathbf{s}(b)) \geq C_G|I| - C''$ for some $C'' > 0$ independent of Z_I . Rearranging, one gets

$$d(b, \mathbf{s}(b)) \geq \frac{C_G d(a, \mathbf{s}(a)) - C'}{r_{\mathfrak{g}}} - C''.$$

Since by definition, for any $a \in U_{X_J, Z_I}^\alpha$,

$$\psi_{X_J, Z_I}^{(\alpha)}(a) = \psi(b)\theta_{X_J, Z_I}^\alpha(a),$$

for some $b \in U_{X_J, Z_I}^\alpha \cap \mathcal{G}_1$, it follows from our assumption $\psi \in \Psi_{\varepsilon; \mathbf{0}}^\infty(\mathcal{G}_1)$ that

$$\psi_{X_J, Z_I}^{(\alpha)}(a) \leq M e^{-\varepsilon' d(b, \mathbf{s}(b))} \leq M' e^{-\varepsilon' C_G r_{\mathfrak{g}}^{-1} d(a, \mathbf{s}(a))},$$

for some $\varepsilon' > \varepsilon$. By the polynomial growth of \mathcal{G} , it follows that $\sum_{X_J, Z_I} \psi_{X_J, Z_I}^{(\alpha)}(a)$ converges uniformly absolutely and satisfies the estimate

$$\sum_{X_J, Z_I} \psi_{X_J, Z_I}^{(\alpha)}(a) \leq M'' e^{-\varepsilon' C_G r_{\mathfrak{g}}^{-1} d(a, \mathbf{s}(a))},$$

for some $M'' > 0$.

We turn to the derivatives of $\psi_{X_J, Z_I}^{(\alpha)}$. Consider $L_{d\tilde{\mathbf{m}}(V\mathbf{r} \oplus W\mathbf{r})} \psi_{X_J, Z_I}^{(\alpha)}$, $V, W \in \Gamma^\infty(\mathbf{A})$. Write $V^{\mathbf{r}}(\mathbf{x}_{X_J, Z_I}^{(\alpha)}(\mu, \tau, x)) = \sum_l v_l(\mu, \tau, x) \partial_l$. Then it follows from definition of $\psi_{X_J, Z_I}^{(\alpha)}$ that

$$(L_{V\mathbf{r}} \psi_{X_J, Z_I}^{(\alpha)})(\mathbf{x}_{X_J, Z_I}^{(\alpha)}(\mu, \tau, x)) = \sum_l v_l(\mu, \tau, x) (\partial_l \psi)(\tau, \mu, P_\alpha(x)).$$

Since $v_l(\mu, \tau, x)|V|^{-1}$ are bounded for all l . The same arguments for the exponential decay as above can be applied.

We turn to $L_{d\tilde{\mathbf{m}}(0 \oplus W\mathbf{r})}(\psi_{X_J, Z_I}^{(\alpha)})$. We use expression (22). By definition, $\text{Supp}(\psi_{X_J, Z_I}^{(\alpha)}) \subseteq \mathbf{s}^{-1}(B(M_1, e^{-\omega r_{\mathfrak{g}}|I|r}))$, therefore it suffices to consider

$$L_{d\tilde{\mathbf{m}}(0 \oplus W\mathbf{r})} \psi_{X_J, Z_I}^{(\alpha)}(\mathbf{x}_{X_J, Z_I}^{(\alpha)}(\tau, \mu, x)), \quad \forall W \in \mathbf{A}|_{B(M_1, e^{-\omega r_{\mathfrak{g}}|I|r})}.$$

Hence Lemma B.7 can be applied to get

$$\begin{aligned} L_{(E_{s_1 Y_1} \circ \dots \circ E_{Z_I} W)^{\mathbf{r}}}(\psi_{X_J, Z_I}^{(\alpha)})(\mathbf{x}_{X_J, Z_I}^{(\alpha)}(\mu, \tau, x)) \\ = e^{N'(\log |I|)^2} \sum_l w_l(\mu, \tau, x)(\partial_l \psi)(\tau, \mu, P_\alpha(x)), \end{aligned}$$

for some functions w_l that are bounded (independent of I).

As for $d(\exp(\mu \cdot (Y_1, \dots, Y_p)) \exp(\tau \cdot (X_1^{\alpha, I}, \dots, X_p^{\alpha, I})) \exp X_J \exp Z_I(x)) \nu(W)(\mathbf{s}(a))$, write $\nu(W) = \sum_l u_l \partial_{x_l}$ on $U^{(\alpha)}$. Then observe that

$$\begin{aligned} d(\exp(\mu \cdot (Y_1, \dots, Y_p)) \exp(\tau \cdot (X_1^{\alpha, I}, \dots, X_p^{\alpha, I})) \exp X_J \exp Z_I(x)) \nu(W) \\ = \sum_l u_l(x) \partial_{x_l} \quad \text{on } U_{X_J, Z_I}^\alpha. \end{aligned}$$

Differentiating $\psi_{X_J, Z_I}^{(\alpha)}$, we get

$$\begin{aligned} \sum_l u_l(x) \partial_{x_l} \psi_{X_J, Z_I}^{(\alpha)}(\mu, \tau, x) \\ = \theta_{X_J}^\alpha \theta_{Z_I}^G \psi(\mathbf{x}_{X_J, Z_I}^{(\alpha)}(\tau, \mu, P^\alpha(x))) \sum_l u_l(x) \partial_{x_l} \theta(2e^{\omega r_{\mathfrak{g}}|I|} r^{-1} \rho(x)). \end{aligned}$$

By Lemma 3.1, and the observation that $x \in B(M_1, e^{-\omega r_{\mathfrak{g}}|I|} r)$, it follows that $\sum_l u_l(x) \frac{\partial}{\partial x_l} \theta(2e^{\omega r_{\mathfrak{g}}|I|} r^{-1} \rho(x))$ is also bounded. Hence we conclude that

$$L_{d\tilde{\mathbf{m}}(V^{\mathbf{r}} \oplus W^{\mathbf{r}})} \psi_{X_J, Z_I}^{(\alpha)}(a) \leq M_1 e^{-\varepsilon'' C_G r_{\mathfrak{g}}^{-1} d(a, \mathbf{s}(a))} (|V| + |W|),$$

for some $\varepsilon'' > \varepsilon$, and similar estimate holds for all derivatives. Therefore $\bar{\psi} \in \Psi_{\varepsilon r_{\mathfrak{g}}^{-1} C_G}^{-\infty}(\mathcal{G})$. \square

Finally, we prove that

Proposition B.9. *Suppose \mathcal{G} is uniformly degenerate. For any $\kappa \in \Psi^{-\infty}(\mathcal{G})$ and differential operator $D \in \Psi^{[m]}(\mathcal{G})$, such that $D|_{\mathcal{G}_k} \bar{\psi} = \kappa|_{\mathcal{G}_k}$,*

$$D\psi - \kappa \in \Psi_{\varepsilon'; \lambda}^{-\infty}(\mathcal{G}).$$

Proof. Replacing κ by an extension of $\kappa|_{M_1}$ similar to $\bar{\psi}$, we may without loss of generality assume $\kappa = 0$.

Consider $\partial_{x_i} L_{d\tilde{\mathbf{m}}(V^{\mathbf{r}} \oplus 0)} \psi_{X_J, Z_I}^{(\alpha)}$ on U_{X_J, Z_I}^α . Recall that $\mathbf{x}_\emptyset^\alpha(\mu', \tau', x')$ is a local coordinates chart around $\mathbf{t} \circ E_{X_J}^\nu E_{Z_I}^\nu(U^\alpha)$. Write $V^{\mathbf{r}} = \sum_{l=1}^n v_l(x') \partial_{(\mu', \tau')_l}$ on $U_\emptyset^{(\alpha)}$. Then $V^{\mathbf{r}}(\mathbf{x}_{X_J, Z_I}^{(\alpha)}(\mu, \tau, x)) = \sum_{l=1}^n u_l(\mu, \tau, E_{X_J}^\nu E_{Z_I}^\nu(x)) \partial_{(\mu, \tau)_l}$. A straightforward calculation gives

$$\begin{aligned} \partial_{x_i} L_{d\tilde{\mathbf{m}}(V^{\mathbf{r}} \oplus 0)} \psi_{X_J, Z_I}^{(\alpha)}(\mu, \tau, x) \\ = \theta(2e^{\omega r_{\mathfrak{g}}|I|} r^{-1} \rho(x)) \sum_{l=1}^n \partial_{x_i} (u_l(\mu, \tau, E_{X_J}^\nu E_{Z_I}^\nu(x))) (\partial_{(\mu, \tau)_l} \theta_{X_J}^\alpha \theta_{Z_I}^G) \psi \\ + (\partial_{x_i} \theta(2e^{\omega r_{\mathfrak{g}}|I|} r^{-1} \rho(x))) \sum_{l=1}^n (u_l(\mu, \tau, E_{X_J}^\nu E_{Z_I}^\nu(x))) (\partial_{(\mu, \tau)_l} \theta_{X_J}^\alpha \theta_{Z_I}^G) \psi. \end{aligned}$$

Since E_Z equals identity on M_1 for all $Z \in \text{Span } \mathbb{R}\{Y_1, \dots, Y_p\}$, it follows that there exists some constants $M > 0$ such that for all $Z \in \text{Span } \mathbb{R}\{Y_1, \dots, Y_p\}$, $|Z| \leq r_{\mathfrak{g}}$,

$$|dE_Z^\nu X| \leq (1 + Md(x, M_1))|X|, \forall X \in T_x M.$$

Iterating, one gets

$$(28) \quad |dE_{Z_I}^\nu \partial_{x_i}(x)| \leq e^{\sum_{i=1}^{|I|} \log(1 + Me^{-\omega r_{\mathfrak{g}}(|I|-i)r})} |\partial_{x_i}(x)|.$$

It is elementary that $\sum_{i=1}^{|I|} \log(1 + Me^{-\omega r_{\mathfrak{g}}(|I|-i)r})$ converges. It follows by integrating $\partial_{x_i} L_{d\tilde{\mathbf{m}}(V^r \oplus 0)} \psi_{X_J, Z_I}^{(\alpha)}(\mu, \tau, x)$ with respect to x_i that

$$\begin{aligned} L_{d\tilde{\mathbf{m}}(V^r \oplus 0)} \psi_{X_J, Z_I}^{(\alpha)}(\mu, \tau, x) &\leq L_{d\tilde{\mathbf{m}}(V^r \oplus 0)} \psi_{X_J, Z_I}^{(\alpha)}(\mu, \tau, P^{(\alpha)}(x)) + M' e^{-\varepsilon' d(\mathbf{x}_{X_J, Z_I}^{(\alpha)}(\mu, \tau, x), x)} \\ &\quad \times (\rho(\mathbf{t}(\mathbf{x}_{X_J, Z_I}^{(\alpha)}(\mu, \tau, x))) + e^{\omega r_{\mathfrak{g}}|I|} \rho(\mathbf{s}(\mathbf{x}_{X_J, Z_I}^{(\alpha)}(\mu, \tau, x)))), \end{aligned}$$

for some constant M' .

The case $\partial_{x_i} L_{d\tilde{\mathbf{m}}(0 \oplus W^r)} \psi_{X_J, Z_I}^{(\alpha)}(\mu, \tau, x)$ is similar. Again write

$$\begin{aligned} d(\exp(\mu \cdot (Y_1, \dots, Y_p)) \exp(\tau \cdot (X_1^{\alpha, I}, \dots, X_p^{\alpha, I})) \exp X_J \exp Z_I(x)) \nu(W) \\ = \sum_l u_l(x) \partial_{x_l} \quad \text{on } U_{X_J, Z_I}^\alpha. \end{aligned}$$

Then one has

$$\begin{aligned} \partial_{x_i} \sum_l u_l(x) \partial_{x_l} \psi_{X_J, Z_I}^{(\alpha)}(\mu, \tau, x) \\ = \psi(\mathbf{x}_{X_J, Z_I}^{(\alpha)}(\tau, \mu, P^\alpha(x))) \partial_{x_i} \left(\theta_{X_J}^\alpha \theta_{Z_I}^G \sum_l u_l(x) \partial_{x_l} \theta(2e^{\omega r_{\mathfrak{g}}|I|} r^{-1} \rho(x)) \right). \end{aligned}$$

It is clear that $\partial_{x_i} (\theta_{X_J}^\alpha \theta_{Z_I}^G \sum_l u_l(x) \partial_{x_l} \theta(2e^{\omega r_{\mathfrak{g}}|I|} r^{-1} \rho(x)))$ is bounded.

As for $\partial_{x_i} E_{X_J} E_{Z_I} W$, for each $Z \in \text{Span } \mathbb{R}\{Y_1, \dots, Y_p\}$, write

$$E_Z \partial_{(\mu', \tau')_l}(x') := \sum_{l'} f_{l'l'}^Z(E_Z^\nu(x')) \partial_{(\mu', \tau')_{l'}}(E_Z^\nu(x')),$$

for some smooth functions $f_{l'l'}^Z$. Then one can express $E_{Z_I} W$ as

$$\begin{aligned} (E_{Z_I} W)^r(\mathbf{x}_\emptyset^{(\alpha)}(\mu', \tau', x')) &= \sum_{l, l_1, l_2, \dots, l_{|I|}, l'} f_{ll_{|I|}}^{Z_{|I|}}(E_{Z_{|I|}}^\nu \cdots E_{Z_1}^\nu(x')) \times \cdots \times f_{ll_1}^{Z_1}(E_{Z_1}^\nu(x')) \\ &\quad \times w_l(x) \partial_{(\mu', \tau')_{l'}}(E_{Z_{|I|}}^\nu \cdots E_{Z_1}^\nu(x')), \end{aligned}$$

where $W^r = \sum_{l=1}^n w_l(x') \partial_{(\mu', \tau')_l}$ and $x' = (E_{Z_{|I|}}^\nu \cdots E_{Z_1}^\nu)^{-1}(x)$. Differentiating with respect to x_i and using the estimates (28) and Lemma B.7, one again obtains

$$\begin{aligned} L_{d\tilde{\mathbf{m}}(0 \oplus W^r)} \psi_{X_J, Z_I}^{(\alpha)}(\mu, \tau, x) &\leq L_{d\tilde{\mathbf{m}}(0 \oplus W^r)} \psi_{X_J, Z_I}^{(\alpha)}(\mu, \tau, P^{(\alpha)}(x)) + M'' e^{-\varepsilon' d(\mathbf{x}_{X_J, Z_I}^{(\alpha)}(\mu, \tau, x), x)} \\ &\quad \times |I| e^{(\log |I|)^2} e^{\omega r_{\mathfrak{g}}|I|} \rho(\mathbf{s}(\mathbf{x}_{X_J, Z_I}^{(\alpha)}(\mu, \tau, x))), \end{aligned}$$

for some constant M'' .

Clearly the same arguments applies for all higher derivatives and one gets similar estimates. Since \mathcal{G} is uniformly degenerate, by choosing U^α to be sufficiently small,

ω can be made sufficiently small. Hence one can sum over all X_J, Z_I and conclude that

$$L_{d\tilde{\mathbf{m}}(V^r \oplus W^r)} D\psi \in \Psi_{\bullet,1}^{-\infty}(\mathcal{G}),$$

for any differential operators D . □

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